Splitting the Classical Model Category Structure on Simplicial Sets

B.Sc. Thesis

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Abstract

In the newly emerging field of homotopy type theory one seeks homotopy-theoretic models for intensional Martin-Löf type theory. An important model is provided by the category **sSet** of simplicial sets where dependent types are interpreted as Kan fibrations. In particular this applies to identity types. However, when it comes to interpreting the eliminator for identity types problems arise since for the fillers whose existence is guaranteed by the axioms of a Quillen model structure it is not obvious how to chose them in a way which is compatible with reindexing. We present the solution given by Vladimir Voevodsky and Thomas Streicher based on type-theoretic universes where one chooses a filler in a slice over a generic context. Since the topos $\mathbf{sSet} = \mathbf{Set}^{\Delta^{\mathrm{op}}}$ of simplicial sets can be considered as a universe within a larger topos $\mathbf{SET}^{\Delta^{\mathrm{op}}}$ this allows us to exhibit a pullback-stable choice of fillers in **sSet**.

We also discuss Voevodsky's Univalence Axiom claiming that isomorphic types (of some universe) are already equal whenever they are isomorphic in a weak sense. This axiom can be formulated in the language of intensional type theory and as shown by Voevodsky in [KLV12] it holds in the simplicial model.

Contents

Introduction

One of the more confusing aspects of intensional Martin-Löf type theory is its distinction between judgmental and propositional equality. The former states equality in the sense of formal rewriting whereas the latter is closer to the mathematical notion of equality in the sense that it can be proved by induction. For example, when defining addition on the natural numbers N as

$$\begin{aligned} \mathsf{add}(n, 0) &= n \\ \mathsf{add}(n, \mathsf{succ}(m)) &= \mathsf{succ}(\mathsf{add}(n, m)) \end{aligned}$$

then although one cannot derive the judgmental equality

$$n : N \vdash \mathsf{add}(0, n) = n : N$$

one can still prove propositional equality in the sense that one can derive

$$n : N \vdash e : \mathrm{Id}_N(\mathsf{add}(0, n), n)$$

for some term e. Here

$$x, y : N \vdash \mathrm{Id}_N(x, y)$$

is a family of types representing the equality predicate on natural numbers.
Generally, for arbitrary types A one can define a family

$$x, y : A \vdash \mathrm{Id}_A(x, y)$$

whose items are referred to as *identity types for A*. Intuitively, the type $\mathrm{Id}_A(a, b)$ is the type of proofs that a is equal to b. Identity types are an instance of inductively defined (families of) types. In the particular case of identity types the introduction and elimination rules are

$$\frac{\Gamma \vdash A}{\Gamma, x{:}A \vdash r_A(x) : \mathrm{Id}_A(x, x)} \text{(Id-I)}$$

$$\frac{\Gamma, x, y{:}A, z{:}\mathrm{Id}_A(x, y) \vdash C(x, y, z) \quad \Gamma, x{:}A \vdash d : C(x, x, r_A(x))}{\Gamma, x, y{:}A, z{:}\mathrm{Id}_A(x, y) \vdash J((x)d)(z) \in C[x, y, z]} \text{(Id-E)}$$

and one postulates the conversion rule

$$J((x)d)(r_A(t)) = d[t/x]$$

specifying how a function defined by the eliminator J is evaluated when applied to a constructor term.

Alas, the following rule

$$\frac{\Gamma \vdash e : \mathrm{Id}_A(x, y)}{\Gamma \vdash x = y \in A} \text{ (Id-R)}$$

identifying propositional and judgmental equality is not derivable in intensional type theory. Adding it gives rise to so-called *extensional type theory* for which, however, type-checking is unfortunately undecidable, cf. [Hof95, Sec. 3.2.2].

Reasoning in intensional type theory can be quite cumbersome since we cannot replace propositionally equal objects in arbitrary contexts. However, using J one can exhibit a term subst such that for $b : B(a)$ and $e : \mathrm{Id}_A(a, a')$ the term $\mathsf{subst}(e, b)$ is of type $B(a')$. As a compensation for this additional bureaucracy one may interpret propositional equality of types A and B in a universe U as being isomorphic in a fairly weak sense. From this point of view it appears as most natural that there are different elements of type $\mathrm{Id}_U(A, B)$ since A and B may be isomorphic in different ways.

Actually, the groupoid model introduced in [HS94, HS98] was motivated by the desire to produce this phenomenon, namely the failure of the Principle of *Uniqueness of Identity Proofs* claiming that any two inhabitants of a given identity type are equal. In this model types are interpreted as groupoids and families of types as fibrations of groupoids. An identity type $\mathrm{Id}_A(x, y)$ is then interpreted by $\mathrm{Hom}_A(x, y)$ and, accordingly, for non-trivial groupoids A the principle of uniqueness of identity proofs fails. The idea of interpreting types as groupoids was motivated by the observation that in intensional type theory one can exhibit operations, namely the proofs of transitivity and symmetry for propositional equality, which endow every type with the structure of an internal groupoid. However, the groupoid laws for these operations hold only in the sense of propositional equality. The proof objects for these propositional equalities can be most naturally understood as 2-cells in the sense of higher-dimensional category theory. Since identity types can be iterated syntax suggests that one actually has n-cells for arbitrary $n \in \mathbb{N}$. Hence, a more faithful picture of syntax is provided when modeling types as higher-dimensional groupoids as in [War11]. But since in syntax the required identities hold only in the sense of propositional equality it appears as even more natural to interpret types as *weak* higher-dimensional groupoids *aka* weak ∞-groupoids. The simplest and oldest notion of a weak ∞-groupoid is given by so-called Kan complexes within the topos **sSet** of simplicial sets, see e.g. [GJ99] for a comprehensive account. In this model families of types are interpreted as Kan fibrations. The simplicial model has been investigated by Vladimir Voevodsky in [KLV12].

When interpreting the eliminator J for identity types in this model one has to choose diagonal fillers for certain square diagrams whose existence is guaranteed by the axioms of Quillen model structures. However, for interpreting syntax these diagonal fillers have to be chosen in such a way that these choices are preserved by (chosen) pullbacks in order to validate the Beck-Chevalley condition for J. In presence of universes this can be achieved by slicing over a generic context and splitting the generic situation once and for all. The particular instances of J are obtained as appropriate pullbacks of this generic solution as described in [Str11a] and [KLV12].

As observed by Voevodsky in [Voe09, KLV12] the simplicial set model validates the so-called *Univalence Axiom* stating that (weakly) isomorphic types of the universe are propositionally equal.

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1 Simplicial Homotopy Theory

We provide the necessary background on model categories and simplicial homotopy theory inspired by [GJ99]. The definition of model categories via weak factorization systems is taken from [AHRT02].

1.1 Model Categories

Definition 1.1.1 (Lifting Properties)**.** In a given category, we consider the following commutative diagram:

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ B & \xrightarrow[g]{} & Y \end{array}$$

A *(diagonal) filler* is a map $h\colon B \to X$ such that the resulting diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle i}\downarrow & {\scriptstyle h}\nearrow & \downarrow{\scriptstyle p} \\ B & \xrightarrow[g]{} & Y \end{array}$$

is commutative. We say i has the *left lifting property with respect to p*. Conversely, p has the *right lifting property with respect to i*.

Definition 1.1.2 (Orthogonality)**.** We write $f \perp g$ and say f *is orthogonal to* g if for every diagram

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ B & \longrightarrow & Y \end{array}$$

there is a (not necessarily unique) diagonal filler.
If $\mathcal{A}$ is a class of morphisms, we define the following morphism classes:

$$\mathcal{A}^{\perp} := \{g\colon f \perp g \text{ for all } f \in \mathcal{A}\}$$
$$^{\perp}\mathcal{A} := \{f\colon f \perp g \text{ for all } g \in \mathcal{A}\}$$

Definition 1.1.3 (Weak Factorization System)**.** A *weak factorization system* in a category is a pair $(\mathcal{L}, \mathcal{R})$ of nonempty classes of morphisms such that:

(i) Every morphism f can be factored as $f = p \circ i$ for $i \in \mathcal{L}$ and $p \in \mathcal{R}$.

(ii) $\mathcal{R} = \mathcal{L}^{\perp}$ and $\mathcal{L} = {}^{\perp}\mathcal{R}$

Definition 1.1.4 (Model Category)**.** A *model category* is a category with finite limits and colimits together with the following classes of morphisms:

- the class $\mathcal{W}$ of *weak equivalences*
- the class $\mathcal{F}$ of *fibrations*
- the class $\mathcal{C}$ of *cofibrations*

The following conditions are required to hold:

(MC1) If two of three maps f, g and $f \circ g$ are weak equivalences, then so is the third (*Two-for-Three-Law*).

(MC2) The pairs $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ and $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ are weak factorization systems.

Elements of $\mathcal{F} \cap \mathcal{W}$ are called *trivial fibrations.* Elements of $\mathcal{C} \cap \mathcal{W}$ are called *trivial cofibrations.*
Instead of "trivial" some authors use the words *anodyne* or *acyclic.*

Every model category has an initial object 0 and a terminal object 1. Hence, the following definition makes sense.

Definition 1.1.5 (Fibrant and Cofibrant Objects)**.** Let $\mathbf{C}$ be a model category. An object $A \in \mathrm{ob}(\mathbf{C})$ is called *fibrant* if $A \to 1$ is a fibration. Dually, A is said to be *cofibrant* if $0 \to A$ is a cofibration.

Example 1.1.6 (Topological Spaces with Serre Fibrations)**.** The category **Top** has the structure of a model category by defining:

(i) A map is a weak equivalence if it is a *weak homotopy equivalence*, i.e. it induces a bijection on the π_0-sets and isomorphisms between the homotopy groups of the respective spaces.

(ii) A map $f \colon X \to Y$ is a fibration if it is a *Serre fibration*, i.e. it has the right lifting property with respect to $\Delta^n \times \{0\} \to \Delta^n \times [0,1]$:

$$\begin{array}{ccc} \Delta^n \times \{0\} & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow f \\ \Delta^n \times [0,1] & \longrightarrow & Y \end{array}$$

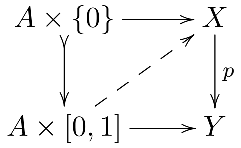
(iii) A map is a cofibration if it has the left-lifting property with respect to trivial fibrations.

Given this structure, every space is fibrant.

Example 1.1.7 (Topological Spaces with Hurewicz Fibrations)**.** The category **Top** can be given another model structure by defining:

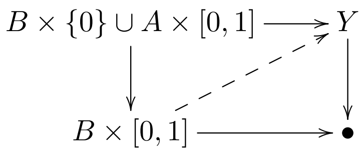
(i) A weak equivalence is a homotopy equivalence.

(ii) A fibration is a *Hurewicz fibration.* This is a map $p\colon X \to Y$ with the homotopy lifting property, i.e. for every space there exists a filler such that the diagram



commutes.

(iii) A cofibration is a *closed Hurewicz fibration.* By this we mean a subspace inclusion $i\colon A \to B$ where A is closed in B and i has the *homotopy extension property.* This means for every space Y there exists a filler as in the following diagram:



Indeed the preceding model structures on **Top** are different in that they yield different morphism classes. For the so-called *Warsaw circle* W the map $W \to \bullet$ is a weak equivalence in the sense of 1.1.6, but not 1.1.7, cf. [DS95, Sec. 3, Ex. 3.6].

1.2 Simplicial Sets

The motivation for simplicial sets is to organize combinatorially defined simplices into sets according to their dimensions and to incidence relations.

Definition 1.2.1 (Simplicial Category)**.** The *simplicial category* Δ contains as objects the finite non-empty ordinals

$$[n] = \{0 \to 1 \to \ldots \to n\}$$

and as morphisms monotone maps between them.

Definition 1.2.2 (Simplical Sets)**.** A *simplicial set* is a functor

$$\Delta^{\mathrm{op}} \to \mathbf{Set},$$

i.e. a set-valued presheaf on the category Δ. The category of simplicial sets with natural transformations between them is denoted by **sSet**. If X is a simplicial set, we write $X_n := X([n])$ and refer to the elements of X_n as the *n-simplices* of X. For $0 \leq k \leq n$, an injective map $[k] \to [n]$ in Δ is called a *non-degenerate k-simplex of* Δ^n.

Definition 1.2.3 (Faces and Degeneracies)**.** For $0 \leq i, j \leq n$ we define the so-called *coface maps*

$$d^i\colon [n-1] \to [n], \quad d^i(0 \to \ldots \to n-1) := (0 \to \ldots \to \hat{i} \to \ldots \to n),$$

where $\hat{i}$ means leaving out the i-th place, and the *codegeneracy maps*

$$s^j\colon [n+1] \to [n], \quad s^j(0 \to \ldots \to n+1) := (0 \to \ldots \to j \xrightarrow{\mathrm{id}_j} j \to \ldots \to n).$$

One can readily verify the following *cosimplicial identities*:

$$\begin{aligned} d^j d^i &= d^j d^{j-1} \quad \text{if } i < j \\ s^j s^i &= s^i s^{j+1} \quad \text{if } i \leq j \\ d^j s^i &= \begin{cases} d^i s^{j-1} & \text{if } i < j \\ \mathrm{id} & \text{if } i \in \{j, j+1\} \\ s^{i-1} d^j & \text{if } j+1 < i \end{cases} \end{aligned}$$

Each morphism in Δ can be written as the product of some d^i precomposed by the product of some s^j, see [Mac98, Ch. VII, 5., Prop. 2]. Hence a simplicial set X can be seen as a graded set $(X_n)_{n\geq 0}$ together with induced morphisms $d_i := X(d^i)$ and $s_j := X(s^j)$ satisfying the *simplicial identites*:

$$\begin{aligned} d_i d_j &= d_{j-1} d_i \quad \text{if } i < j \\ s_i s_j &= s_{j+1} s_i \quad \text{if } i \leq j \\ s_i d_j &= \begin{cases} s_{j-1} d_i & \text{if } i < j \\ \mathrm{id} & \text{if } i \in \{j, j+1\} \\ d_j s_{i-1} & \text{if } j+1 < i \end{cases} \end{aligned}$$

Definition 1.2.4 (Standard n-Simplex in **sSet**)**.** In the category of simplicial sets **sSet**, the standard n-simplex Δ^n is the contravariant functor represented by $[n]$, i.e.

$$\Delta^n := \mathrm{Hom}_\Delta(-, [n]).$$

In particular, the 0-simplex is the terminal object $\Delta^0 = \bullet$.

Remark 1.2.5. By the Yoneda Lemma, n-simplices of Y are classified by simplicial maps $\Delta^n \to Y$. There is a natural isomorphism

$$\mathrm{Hom}_{\mathbf{sSet}}(\Delta^n, Y) \cong Y_n.$$

Definition 1.2.6 (Boundary and Horn)**.** The *boundary* of Δ^n, written $\partial\Delta^n$, is the subcomplex generated by the $(n-1)$-faces of the standard simplex, i.e.

$$\partial\Delta^n := \bigcup_{i=0}^{n} \partial^i \Delta^n$$

with $\partial^i\Delta^n := \mathbf{y}(d^i)$. More explicitly

$$\partial\Delta^n([m]) = \{\alpha\colon [m] \to [n] : \alpha \text{ is not an epimorphism}\}.$$

The *k-th horn of* Δ^n for $0 \leq k \leq n$ is the subcomplex Λ^n_k generated by all such faces except the k-th one, i.e.

$$\Lambda^n_k := \bigcup_{i \neq k} \partial^i\Delta^n.$$

1.2.1 Geometric Realization

Definition 1.2.7 (Standard n-Simplex in **Top**)**.** The *standard n-simplex functor* is

$$|-| : \Delta \to \mathbf{Top}$$

where

$$|[n]| := \left\{ \sum_{i=0}^{n} t_i e_i \in \mathbb{R}^{n+1} \colon t_i \geq 0, \;\; \sum_{i=0}^{n} t_i = 1 \right\}$$

and a morphism $\alpha\colon [n] \to [m]$ is mapped to

$$|\alpha|\colon \; |[n]| \to |[m]|, \quad \sum_{i=0}^{n} t_i e_i \mapsto \sum_{i=0}^{n} t_i e_{\alpha(i)}.$$

Definition 1.2.8 (Singular Complex and Singular Functor)**.** Let X be a topological space. The simplicial set

$$S(X)\colon \Delta^{\mathrm{op}} \to \mathbf{Set}, \quad [n] \mapsto \mathrm{Hom}_{\mathbf{Top}}(|\Delta^n|, X)$$

is called *singular complex* $S(X)$. This yields a functor

$$S\colon \mathbf{Top} \to \mathbf{sSet}, \quad X \mapsto S(X),$$

called the *singular functor*.

In order to turn a simplicial set into a geometric object, we want to define a functor $|-| : \mathbf{sSet} \to \mathbf{Top}$, called geometric realization, which extends the standard n-simplex functor. It will be given as the left-adjoint to the singular functor.

Definition 1.2.9 (Geometric Realization)**.** The *geometric realization* of a simplicial set X is given as the colimit

$$|X| := \operatorname*{colim}_{\substack{\Delta^n \to X \\ \text{in } \Delta/X}} |\Delta^n|$$

in the category of topological spaces.

Lemma 1.2.10. *Let X be a simplicial set. There is an isomorphism*

$$X \cong \operatorname*{colim}_{\substack{\Delta^n \to X \\ in\ \Delta/X}} \Delta^n = \operatorname*{colim}_{\substack{\Delta^n \to X \\ in\ \Delta/X}} \mathrm{Hom}_{\Delta}(-, [n]).$$

Proof. By [MM92, Ch. 1, 5., Prop. 1] any presheaf $\mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$ is the colimit of a family of representable objects if $\mathbf{C}$ is a small category. By the proof given in the reference, in our situation, X turns out to be the colimit of the diagram

$$\mathrm{Elts}(X) \xrightarrow{\pi} \Delta \xrightarrow{\mathrm{Hom}_\Delta(-,[n])} \mathbf{Set}^{\Delta^{\mathrm{op}}}. \tag{1.1}$$

where $\pi\colon \mathrm{Elts}(X) \to \Delta$ denotes the projection from the category of elements. By Remark 1.2.5 any element in X_n can be canonically regarded as a morphism $\Delta^n \to X$ in Δ/X, hence, in (1.1) we may as well take the colimit over the morphisms of the category Δ/X. □

Theorem 1.2.11 (Adjunction $|-| \dashv S$). *The geometric realization functor is left adjoint to the singular spaces functor, i.e. there is an isomorphism*

$$\mathrm{Hom}_{\mathbf{Top}}(|X|, Y) \cong \mathrm{Hom}_{\mathbf{sSet}}(X, SY)$$

natural in both simplicial sets X and topological spaces Y.

Proof. By the cocontinuity of contravariant representable functors we have

$$\mathrm{Hom}_{\mathbf{Top}}(|X|, Y) \cong \operatorname*{colim}_{\substack{\Delta^n \to X \\ \text{in } \Delta/X}} \mathrm{Hom}_{\mathbf{Top}}(|\Delta^n|, Y) = \operatorname*{colim}_{\substack{\Delta^n \to X \\ \text{in } \Delta/X}} SY([n]).$$

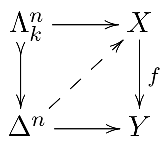
Remark 1.2.5 yields that this is isomorphic to $\operatorname{colim}_{\substack{\Delta^n \to X \\ \text{in } \Delta/X}} \mathrm{Hom}_{\mathbf{sSet}}(\Delta^n, SY)$ which, again by cocontinuity is isomorphic to $\mathrm{Hom}_{\mathbf{sSet}}(X, SY)$. □

Theorem 1.2.12. *For each simplicial set, the geometric realization $|X|$ is a CW-complex.*

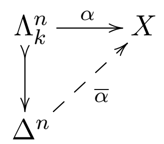
For a proof cf. [GJ99, Ch. I, 2., Prop 2.3].

1.2.2 The Model Structure on the Category of Simplicial Sets

Definition 1.2.13 (Kan Fibration)**.** A morphism of simplicial sets $f\colon X \to Y$ is called a *Kan fibration* if every commuting square indicated as follows has a diagonal filler:



Definition 1.2.14 (Kan Complex)**.** A simplicial set X is a *Kan complex* if it is fibrant with respect to Kan fibrations, i.e. every morphism $\alpha\colon \Lambda^n_k \to X$ can be extended to a map $\overline{\alpha}\colon \Delta^n \to X$ in the following sense:



Definition 1.2.15 (Weak equivalences)**.** A morphism $f\colon X \to Y$ is a *weak equivalence* if the induced map $|f| : |X| \to |Y|$ is a *homotopy equivalence*, i.e. there is a continuous map $\varphi\colon |Y| \to |X|$ such that $|f| \circ \varphi \simeq \mathrm{id}_{|Y|}$ and $\varphi \circ |f| \simeq \mathrm{id}_{|X|}$.

It is a deep theorem that this defines the structure of a model category in **sSet**:

Theorem 1.2.16. *The category* **sSet** *becomes a model category by defining*

- *weak equivalences to be weak equivalences as defined above,*
- *fibrations to be Kan fibrations,*
- *cofibrations to be monomorphisms.*

A proof can be found in [GJ99, Ch. I.11, Thm. 11.3].

1.2.3 Slice Categories and Right-Properness

Theorem 1.2.17 (Stability under Slicing)**.** *Let* $\mathbf{C}$ *be a model category and* $X \in \mathrm{ob}(\mathbf{C})$*. We define a map in* $\mathbf{C}/X$ *to be a fibration, cofibration or weak equivalence, resp., if the underlying map in* $\mathbf{C}$ *is a fibration, cofibration or weak equivalence, resp. This gives rise to a model category structure on* $\mathbf{C}/X$*.*

Definition 1.2.18. A model category is called *right proper* iff the class of weak equivalences is closed under pullbacks along fibrations.

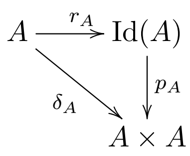
Theorem 1.2.19. *If* $\mathbf{C}$ *is a right proper model category, then so is* $\mathbf{C}/X$ *for any object* X *in* $\mathbf{C}$*.*

Indeed, **sSet** is an instance of a right-proper model category.
For further details see [AK12, Sec. 3].

2 Interpreting Martin-Löf Type Theory in a Universe in sSet

2.1 Interpreting Types and Identity Types in sSet

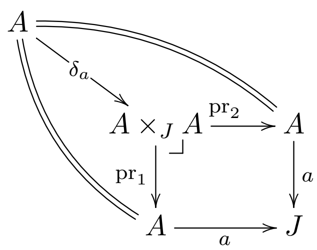
The judgment $\vdash A$ that A is a type is interpreted by a fibrant object A in **sSet**. A context $\Gamma \vdash A$ is interpreted as a fibration having the interpretation of Γ as the codomain. A term $\Gamma \vdash a : A$ in a context Γ is interpreted as a section of the interpretation of $\Gamma \vdash A$. Following Steve Awodey and Michael Warren in [AW09], the identity type $\mathrm{Id}(A)$ on a type A is interpreted by the *path object* $\mathrm{Id}(A) := A^I$ for $I := \Delta^1$. This yields a factorization of the diagonal $\delta_A \colon A \to A \times A$ as



where $r_A \colon A \to A^I$ is obtained by transposition from the projection $I \times A \to A$, and the map $p_A \colon A^I \to A^{\partial I} \cong A \times A$ is induced by the inclusion $\partial I \to I$.

2.2 Lifting Grothendieck Universes to Type-Theoretic Universes

According to our interpretation, a family of types is given as a Kan fibration $a \colon A \to J$. We want to interpret the respective identity type in an analogous way by factoring the *fiberwise diagonal* $\delta_a \colon A \to A \times_J A$ defined by:



But these factorizations are in general not stable under change of base, i.e. under a choice of pullbacks along some $u \colon J \to I$. Explicating the ideas in [Str04], [Str11a] and [HS99], we want to solve this problem by introducing type-theoretic universes in the

sense of Martin-Löf. These universes are *split* in the sense that they admit a functorial choice of fillers of square diagrams that is consistent with change of base.
To construct a universe inside a presheaf topos let $\mathcal{U}$ be a Grothendieck universe in **Set**. If $\mathbf{C}$ is a category internal to $\mathcal{U}$, this gives rise to a morphism $p_U \colon \tilde{U} \to U$ in $\widehat{\mathbf{C}} = \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$, serving as a type-theoretic universe, by defining

$$U(I) := \mathcal{U}^{(\mathbf{C}/I)^{\mathrm{op}}}, \quad U(\alpha) := \mathcal{U}^{\Sigma_\alpha^{\mathrm{op}}}$$

where for $\alpha \colon J \to I$ the functor $\Sigma_\alpha \colon \mathbf{C}/J \to \mathbf{C}/I$ is postcomposition with α. We then let

$$\tilde{U}(I) := \{\langle A, a\rangle \colon A \in U(I),\ a \in A(\mathrm{id}_I)\}$$

and

$$\tilde{U}(\alpha)(\langle A, a\rangle) = \langle U(\alpha)(A), A(\alpha \xrightarrow{\alpha} \mathrm{id}_I)(a)\rangle$$

for $\alpha \colon J \to I$ in $\mathbf{C}$.

Definition 2.2.1 ($\mathcal{U}$-small Family)**.** Let $f \colon B \to A$ be a presheaf morphism in $\widehat{\mathbf{C}}$ and $\mathcal{U}$ a Grothendieck universe in **Set**. Then f is called *$\mathcal{U}$-small* if for all $I \in \mathrm{ob}(\mathbf{C})$ and $a \in A(I)$ the fiber $f_I^{-1}(a)$ is isomorphic to some set in $\mathcal{U}$.

Definition 2.2.2 (Genericity)**.** Let $\mathcal{A}$ be a class of morphisms in $\widehat{\mathbf{C}}$. A morphism f in $\mathcal{A}$ is said to be *generic* or *universal for* $\mathcal{A}$ if every morphism in $\mathcal{A}$ can be obtained as a pullback of f along some morphism in $\widehat{\mathbf{C}}$.

Theorem 2.2.3. *The morphism p_U is generic for $\mathcal{U}$-small maps.*

For the proof, cf. [Str04, Sec. 3].

2.3 Splitting the Classical Model Category Structure on Simplical Sets

2.3.1 The Generic Universe $\tilde{U}$

As we want to model type theory in simplicial sets we have to consider the case $\mathbf{C} = \Delta$. We adapt the idea of the previous subsection, but restrict the universe to Kan fibrations. First, note that for $A \in \widehat{\mathbf{C}}$ we have

$$\widehat{\mathrm{Elts}(A)} \simeq \widehat{\mathbf{C}}/A$$

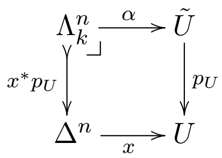
where $\mathrm{Elts}(A) = \mathbf{y}_{\mathbf{C}} \downarrow A$. Now, the category of $\mathcal{U}$-valued presheaves $\mathcal{U}^{(\mathbf{C}/I)^{\mathrm{op}}}$ is equivalent to the full subcategory of $\widehat{\mathbf{C}}/\mathbf{y}(I)$ of morphisms with $\mathcal{U}$-small fibers.
By the Grothendieck construction, for $A \in \mathcal{U}^{(\Delta/[n])^{\mathrm{op}}}$ there is a corresponding morphism of simplicial sets $P_A \colon \mathrm{Elts}(A) \to \Delta^n$.
We then define

$$U([n]) := \left\{ A \in \mathcal{U}^{(\Delta/[n])^{\mathrm{op}}} \colon P_A \text{ is a Kan fibration} \right\}.$$

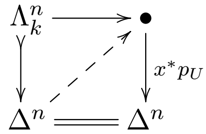
As Kan fibrations are stable under pullbacks, the newly defined U is a sub-presheaf of the old one. It also follows that for morphisms α in Δ we can define $\tilde{U}$ and p_U as we did before, but restricted to our newly constructed U.

Theorem 2.3.1. *The class $\mathcal{U}$ defined above is a universe in $\widehat{\mathbf{C}}$, i.e. $p_U \colon \tilde{U} \to U$ is a Kan fibration and generic for $\mathcal{U}$-small Kan fibrations.*

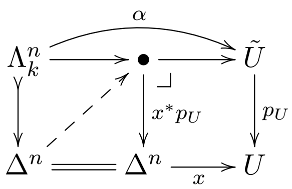
Proof. To show that p_U is a Kan fibration we have to find a diagonal filler for the diagram:



By definition of p_U, the pullback $x^* p_U \colon \bullet \to \Delta^n$ is a Kan fibration, hence we have a diagonal filler in:



This gives rise to a diagram



yielding the desired filler $\Delta^n \to \tilde{U}$.
Now, let $a \colon A \to I$ be a $\mathcal{U}$-small Kan fibration. Then, by the definition of p_U we have that a is the pullback of p_U along some morphism that sends a generalized element $x \colon \Delta^n \to I$ to an element of $\mathcal{U}^{(\Delta/[n])^{\mathrm{op}}}$ corresponding to $x^* a$. $\square$

We want to show that the simplicial set U is a Kan complex. This has been proved by André Joyal in the note [Joy11] and Chris Kapulkin, Peter Lumsdaine and Vladimir Voevodsky in [KLV12].
See [May67, Ch. II, Sec. 10] for the following definition needed for technical reasons.

Definition 2.3.2 (Minimal Kan fibration)**.** A Kan fibration $f \colon X \to Y$ is called *minimal* if for each $n \in \mathbb{N}$ and $x, x' \in X_n$ the conditions $f(x) = f(x')$ and $d_i(x) = d_i(x')$, $i \neq k$, imply $d_k(x) = d_k(x')$.

Theorem 2.3.3 (Quillen's Lemma)**.** *Let $f \colon Y \to X$ be a fibration. Then there is a factorization $f = p \circ g$, where p is a minimal fibration and g is a trivial cofibration.*

Lemma 2.3.4. *Let X be a contractible simplicial set, i.e. $|X|$ is a contractible topological space. If $x_0 \in X$ and $p\colon Y \to X$ is a minimal fibration with fiber $F := Y_{x_0}$, then there is an isomorphism:*

$$\begin{tikzcd} Y \arrow[r, "\cong"] \arrow[d, "p"'] & F \times X \arrow[dl, "\mathrm{pr}_2"] \\ X & \end{tikzcd}$$

Lemma 2.3.5. *Let $f\colon A \to B$ be a cofibration and consider the induced pullback $f^*\colon \mathbf{sSet}/B \to \mathbf{sSet}/A$. In the chain of adjoints $\Sigma_f \dashv f^* \dashv \Pi_f$ we have:*

(i) The adjoint $\Pi_f\colon \mathbf{sSet}/A \to \mathbf{sSet}/B$ preserves trivial fibrations.

(ii) The counit $f^*\Pi_f \to \mathrm{id}_{\mathbf{sSet}/A}$ is an isomorphism.

(iii) If $p\colon E \to A$ is $\mathcal{U}$-small, then so is $\Pi_f p$.

Proof. (i) Let $p\colon X \to Y$ be a trivial fibration and $g\colon V \to W$ a cofibration. As $\mathcal{C}$ is stable under pullbacks, any diagram of the form

$$\begin{tikzcd} f^*V \arrow[r] \arrow[d, tail, "f^*g"'] & X \arrow[d, "p"] \\ f^*W \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

has a diagonal filler. By transposition there is also a diagonal filler in:

$$\begin{tikzcd} V \arrow[r] \arrow[d, "g"'] & \Pi_f X \arrow[d, "\Pi_f p"] \\ W \arrow[r] \arrow[ur, dashed] & \Pi_f Y \end{tikzcd}$$

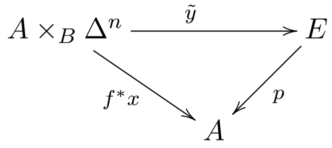
This means that $\Pi_f p$ is a trivial fibration.

(ii) Since f is a monomorphism, the functor Σ_f with $\Sigma_f(p) = f \circ p$ is full and faithful. Hence, the unit $\mathrm{id}_{\mathbf{sSet}/A} \to f^*\Sigma_f$ is an isomorphism, and by transposition $f^*\Pi_f \to \mathrm{id}_{\mathbf{sSet}/A}$ as well (cf. [Mac98, Ch. VII, Sec. 4, Lem. 1]).

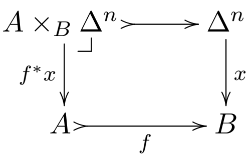
(iii) Let $x\colon \Delta^n \to B$ be an n-simplex. An element of the fiber $(\Pi_f p)^{-1}(x)$ is an n-simplex y of $\Pi_f E$ such that the diagram

$$\begin{tikzcd} \Delta^n \arrow[rr, "y"] \arrow[dr, "x"'] & & \Pi_f E \arrow[dl, "\Pi_f p"] \\ & B & \end{tikzcd}$$

commutes. By transposition, y corresponds to $\tilde{y}$ such that the diagram



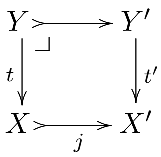
commutes. Hence $(\Pi_f p)^{-1}(x) \cong \mathrm{Hom}_{\mathbf{sSet}/B}(x, \Pi_f p) \cong \mathrm{Hom}_{\mathbf{sSet}/A}(f^*x, p)$. Considering the pullback diagram



we see that the pullback $A \times_B \Delta^n$ is a subobject of Δ^n, hence contains only finitely many non-degenerate simplices. Thus, $f^*x\colon A \times_B \Delta^n \to A$ injects into finitely many fibers of p and consequently is $\mathcal{U}$-small.

□

Theorem 2.3.6 (Joyal)**.** *Let $t\colon Y \to X$ be a trivial fibration and $j\colon X \to X'$ a cofibration. Then there exists a trivial fibration $t'\colon Y' \to X'$ such that*



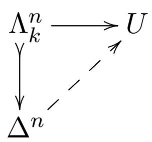
is a pullback. If t is $\mathcal{U}$-small, then t' can be chosen $\mathcal{U}$-small as well.

Proof. Let $t' := j_* t$. By Lemma 2.3.5(i), t' is a trivial fibration. Because of (ii), we have $j^*t' \cong t$, and (iii) implies $\mathcal{U}$-smallness. □

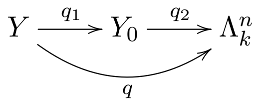
We are now able to prove the desired result.

Theorem 2.3.7. *The simplicial set U is a Kan complex.*

Proof. We have to show that we can extend any horn in U to a simplex:



By construction of p_U such a horn corresponds to a $\mathcal{U}$-small fibration $q\colon Y \to \Lambda^n_k$. Quillen's Lemma yields a factorization of q as



where q_1 is a trivial fibration and q_2 a minimal fibration. Both are still $\mathcal{U}$-small: Since a trivial fibration is surjective, each fiber of q_2 is a quotient of a fiber of q, and thus small. By Lemma 2.3.4, there is an isomorphism $Y_0 \cong F \times \Lambda^n_k$, so there is a pullback:

$$\begin{array}{ccc} Y_0 & \rightarrowtail & F \times \Delta^n \\ {\scriptstyle q_2}\downarrow & \lrcorner & \downarrow \\ \Lambda^n_k & \rightarrowtail & \Delta^n \end{array}$$

Applying Joyal's Theorem 2.3.6 to the trivial fibration q_1, we get a trivial fibration $Y' \to F \times \Delta^n$ making the diagram

$$\begin{array}{ccc} Y & \longrightarrow & Y' \\ {\scriptstyle q_1}\downarrow & \lrcorner & \downarrow \\ Y_0 & \rightarrowtail & F \times \Delta^n \end{array}$$

commute. Finally, we can glue the diagrams as follows:

$$\begin{array}{ccc} Y & \longrightarrow & Y' \\ {\scriptstyle q_1}\downarrow & \lrcorner & \downarrow \\ Y_0 & \rightarrowtail & F \times \Delta^n \\ {\scriptstyle q_2}\downarrow & \lrcorner & \downarrow \\ \Lambda^n_k & \rightarrowtail & \Delta^n \end{array}$$

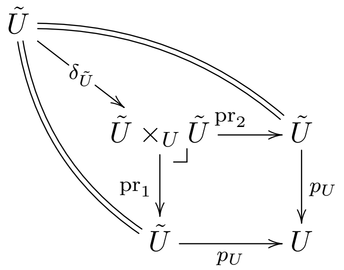
Since q_1 and q_2 are $\mathcal{U}$-small, by Theorem 2.3.6, the maps $Y' \to F \times \Delta^n$ and $F \times \Delta^n \to \Delta^n$ are $\mathcal{U}$-small fibrations, and so is their composite. □

2.3.2 Interpreting Dependent Sums and Products in $\tilde{U}$

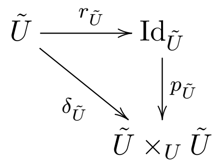
For a fibration $f \colon B \to A$, the dependent product along f is interpreted by Π_f and the dependent sum is interpreted by Σ_f, respectively. Since types are interpreted by Kan complexes, Π_f and Σ_f are required to preserve fibrations. The statement is clear for $\Sigma_f = f \circ (-)$ since the class of fibrations is closed under composition. Since $f^* \dashv \Pi_f$ is an adjunction between model categories, Π_f preserves fibrations and anodyne fibrations if and only if f^* preserves cofibrations and trivial cofibrations. The latter is true as **sSet** is right-proper.

2.3.3 Interpreting Identity Types in $\tilde{U}$

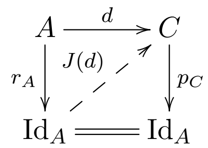
To interpret identity types in this universe, as in [AW09] we consider the fiberwise diagonal $\delta_{\tilde{U}} \colon \tilde{U} \to \tilde{U} \times_U \tilde{U}$ over the pullback



and a factorization



where $p_{\tilde{U}}$ is a fibration and $r_{\tilde{U}}$ is a trivial cofibration.
Let I be a simplicial set. If $p_C \colon C \to \mathrm{Id}_A$ is a fibration in $\mathbf{sSet}/I$ and $d \colon A \to C$ a morphism in $\mathbf{sSet}/I$ with $p_C \circ d = r_A$ then one can choose $J(d)$ in $\mathbf{sSet}/I$ such that



commutes since $(\mathcal{C} \cap \mathcal{W})^{\perp} = \mathcal{F}$ (cf. [AW09]). Now, for any square in a slice we can still find such a filler $J(d)$. But given some $u \colon K \to I$, in general u^* will not preserve this choice of fillers. In other words, the Beck-Chevalley condition for K does not hold in general. Using the methods introduced in the previous paragraphs, we will split the model category structure once and for all to obtain a consistent choice of fillers.
The eliminator J arises in the elimination rule for identity types. To interpret J, we pull back the diagram along the projection p, which goes from the generic context

$$A : U, C : (x, y : A)U^{\mathrm{Id}_A(x,y)}, d : (x : A)C(x, x, r_A(x))$$

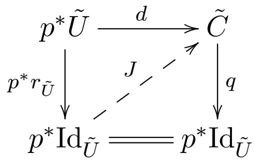
to U. As p is a Kan fibration and pullbacks along Kan fibrations preserve weak equivalences, also $p^* r_{\tilde{U}}$ is an anodyne cofibration.
Let $q \colon \tilde{C} \to p^* \mathrm{Id}_{\tilde{U}}$ be the interpretation of the type family

$$\Gamma, x, y : A, z : \mathrm{Id}_A(x, y, z) \vdash C(x, y, z)$$

and $d \colon p^* \tilde{U} \to \tilde{C}$ be the interpretation of the term

$$\Gamma, x, y : A, z : \mathrm{Id}_A(x, y, z) \vdash d(x) : C(x, y, z).$$

Then we have the factorization $p^*r_{\tilde{U}} = q \circ d$. As q is a Kan fibration and $p^*r_{\tilde{U}}$ a trivial cofibration, the model category structure of **sSet** yields a map J such that the diagram



commutes.
Let $\Delta \dashv \Gamma : \mathbf{sSet} \to \mathbf{Set}$ where $\Gamma = \mathrm{Hom}_{\mathbf{sSet}}(1, -)$ is the global sections functor. All discrete simplicial sets $\Delta(S) = \coprod_S 1$ are Kan complexes and all $\Delta(f)$ are Kan fibrations. Hence, **sSet** contains **Set** as a submodel. Just as we considered a Grothendieck universe $\mathcal{U}$ in **Set**, we can consider **Set** as a Grothendieck universe in a larger model **SET** of set theory. Thus, **Set** gives rise to a universe $p_U \colon \tilde{U} \to U$ in $\mathbf{sSET} := \mathbf{SET}^{\Delta^{\mathrm{op}}}$ for which we have a split choice of J by the above considerations. This allows one to interpret J in **sSet** in such a way that it is stable under the choice of pullbacks given by U.

3 The Univalence Axiom in the Category of Simplical Sets

In this section we briefly discuss Voevodsky's Univalence Axiom on which more detailed information can be found in [Str11b, KLV12, PW12].

3.1 The Univalence Axiom

Let X and Y be types in a universe U. In addition to the identity type $\mathrm{Id}_U(X,Y)$ one can define a type $\mathsf{Weq}(X,Y:U)$ of *weak equivalences from* X *to* Y in the following way:

$$\begin{aligned}
&\mathsf{isContr}(X:U) = (\Sigma x : X)(\Pi y : Y)\mathrm{Id}_X(x,y)\\
&\mathsf{hFiber}(X,Y:U)(f:X\to Y)(y:Y) = (\Sigma x:X)\mathrm{Id}_Y(f(x),y)\\
&\mathsf{isWeq}(X,Y:U)(f:X\to Y) = (\Pi y:Y)\mathsf{isContr}(\mathsf{hFiber}(X,Y,f,y))\\
&\mathsf{Weq}(X,Y:U) = (\Sigma f\colon X\to Y)\mathsf{isWeq}(X,Y,f)
\end{aligned}$$

Using J one can define a map

$$\mathsf{eqWeq}(X,Y:U)\colon \mathrm{Id}_U(X,Y)\to \mathsf{Weq}(X,Y:U)$$

sending $r_U(X) : \mathrm{Id}_U(X,X)$ to the weak equivalence $\mathrm{id}_X : \mathsf{Weq}(X,X:U)$. Now we can formulate Voevodsky's Univalence Axiom.

Univalence Axiom. *The map* $\mathsf{eqWeq}(X,Y:U)\colon \mathrm{Id}_U(X,Y)\to\mathsf{Weq}(X,Y:U)$ *is a weak equivalence, i.e. one postulates a constant*

$$\mathsf{UA} : (\Pi X,Y:U)\mathsf{isWeq}(\mathsf{eqWeq}(X,Y:U)).$$

As shown e.g. in [PW12] the proposition $\mathsf{isWeq}(X,Y)(f)$ is equivalent to the requirement that f be a weak isomorphism meaning that there is a map $g\colon Y\to X$ such that $g\circ f$ and $f\circ g$ are pointwise equal to the respective identities, i.e.

$$\mathsf{isIso}(X,Y)(f) = (\Sigma g : Y\to X)\,((\Pi x:X)\mathrm{Id}_X(g(f(x)),x))\times((\Pi y:Y)\mathrm{Id}_Y(f(g(y)),y))\,.$$

In the simplicial set model the proposition $\mathsf{isIso}(X,Y)(f)$ expresses that f is a homotopy equivalence. This is in accordance with the fact that maps between Kan complexes are weak equivalences if and only if they are homotopy equivalences.

As an alternative and convenient reformulation of the Univalence Axiom one may postulate a constant $\mathsf{WeqIndElim}$ of type

$$\begin{aligned}&(\Pi C : (\Pi X, Y : U))\mathsf{Weq}(X, Y) \to U)\\&(\Pi d : (\Pi X : U))C(X, X, \mathsf{eqWeq}(r_U(X)))\\&(\Pi X, Y : U)(\Pi e : \mathsf{Weq}(X, Y))C(X, Y, e)\end{aligned}$$

which allows one to prove a property for all weak equivalences by showing it just for those of the form $\mathsf{eqWeq}(r_U(X))$.
This provides a most useful *induction principle for weak equivalences.*
The *Principle of Function Extensionality*

$$((\Pi x : X)\mathrm{Id}_Y(f(x), g(x)) \to \mathrm{Id}_{X\to Y}(f, g))$$

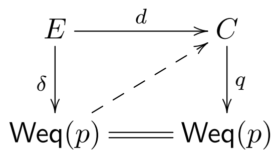
is not derivable in intensional type theory (hence the epitheton "intensional") but it can be derived from the Univalence Axiom as shown by Nicola Gambino in [Gam11].

3.2 The Univalence Axiom in the Category of Simplicial Sets

Indeed, the Univalence Axiom holds in **sSet**. This has been proven by Chris Kapulkin, Peter Lumsdaine and Vladimir Voevodsky in [KLV12] as well as Ieke Moerdijk in [Moe11]. We do not give details but formulate explicitly the categorical property which has to be proved in order to guarantee the validity of the Univalence Axiom. Let B be a simplicial set. For morphisms $p\colon E \to B$ and $p'\colon E' \to B$ the exponential in $\mathbf{sSet}/B$ is given by

$$\hom_B(p, p')(b) = \mathrm{Hom}_{\mathbf{sSet}}(b^*p, b^*p')$$

where $b\colon \Delta^n \to B$ is an an n-simplex of B. For Kan fibrations p and p' one can show that $\hom_B(p, p') \to B$ is a Kan fibration. In this case one may consider the subobject $\mathsf{Weq}_B(p, p')$ of $\hom_B(p, p')$ consisting of those $w\colon b^*p \to b^*p'$ which are weak equivalences. As shown in [KLV12] the map $\mathsf{Weq}_B(p, p') \to B$ is again a Kan fibration.
For a Kan fibration p let $\mathsf{Weq}(p) := \mathsf{Weq}_{B\times B}(\mathrm{pr}_1^*p, \mathrm{pr}_2^*p)$. We consider the diagonal map $\delta\colon p \to \mathsf{Weq}(p)$ over $B \times B$ sending a simplex $b\colon \Delta^n \to B$ to the identity on b^*p. Now a fibration $p\colon E \to B$ is called *univalent* if $\delta\colon p \to \mathsf{Weq}(p)$ is a weak equivalence. In other words, univalence claims that $\delta\colon p \to \mathsf{Weq}(p)$ is a trivial cofibration which is equivalent to δ having the left lifting property with respect to Kan fibrations. Since Kan fibrations are stable under arbitrary pullbacks univalence is equivalent to the requirement that any square



with a Kan fibration $q\colon C \to \mathsf{Weq}(p)$ has a diagonal filler. This condition expresses the validity of the aforementioned induction principle for weak equivalences.
The name "univalence" may be motivated by the fact that it expresses the injectivity of the family

$$A : U \vdash U$$

in the sense that weakly isomorphic types of U are already propositionally equal.

3.3 Synthetic Homotopy Theory

One of the most promising aspects of homotopy type theory is to use the language of type theory for developing homotopy theory in a "synthetic" way. This means that instead of explicitly constructing path spaces in some concrete model like **Top** or **sSet** one postulates the existence of identity types. A most readable introduction to this aspect can be found in [PW12] where the development is accompanied by Coq code illustrating how this endeavor can benefit from using an interactive theorem prover based on type theory. For example in Coq it is shown that a map between types in the universe is a weak equivalence if and only if it is a homotopy equivalence. An even more immediate example is the verification that for every $A : U$ and $a : A$ the type $\mathrm{Id}_A(a, a)$ is a group, the so-called *loop space*.
Using the language of type theory one can define *homotopy levels* or *h-levels* as predicates on the universe by induction over the natural numbers. Types of h-level 0 are the contractible spaces and types of h-level $n + 1$ are those types A for which all identity types $\mathrm{Id}_A(a, b)$ are of h-level n. One easily proves by induction that a type of h-level n is also of h-level $n + 1$.
For example a type A is of h-level 1 if all identity types $\mathrm{Id}_A(a, b)$ are contractible, i.e. all elements of A are propositionally equal and all these equality propositions are contractible. For this reason in [PW12] types of h-level 1 are called *h-propositions*. Families of h-propositions are weakly classified by the map $i_1\colon 1 \to 1 + 1$. Types of h-level 2 are all those types all whose identity types are h-propositions for which reason they are called *h-sets* in [PW12]. It is easy to show that types of h-level n are closed under dependent products of families indexed by types in U.
The language of type theory does not allow one to define a basic geometric object such as spheres. To overcome this limitation there have been considered so-called *higher inductive types*, cf. [Shu12]. However, it is not clear so far to which extent these new forms of inductive definition can be given a computational meaning.

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