The Cubical Model of Type Theory

M.Sc. Thesis

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Abstract

Recently, a model for dependent type theory has been given by Coquand et al. in a variant of the category of cubical sets. In their interpretation, they give a constructive version of Kan filling, classically known from simplicial sets, by means of a newly introduced operation. This cubical model also validates the univalence axiom constructively.
We present this model in terms of classical Kripke-Joyal semantics in a topos.

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Introduction

Proof-Theoretic Aspects of Dependent Type Theory

In dependent type theory the paradigm of *propositions-as-types* plays a central role. This principle is implemented in the type theory introduced by Per Martin-Löf and Jean-Yves Girard in the early 1970s. Propositions-as-types essentially allows for proofs as objects, which are captured by terms in the language. Hence, a proposition is considered as the *type of its proofs*. These types are called *propositional types* and constitute a constructive interpretation of logic.

A proof of $A \implies B$ is interpreted by a function mapping proofs of A to proofs of B. Universal quantification corresponds to products of families of propositional types, whereas existential quantification corresponds to disjoint unions or sums of families of propositional types.

Due to the constructiveness of the logic of type theory, proof objects carry algorithmic information so that functional programs can be extracted from the proofs. Let A and B be sets of inputs and outputs, respectively, and $P \in (A \to B \to \mathsf{Set})$ be a predicate specifying the relation between inputs and outputs. Assume we are given a proof object $p \in (\Pi x : A)(\Sigma y : B)P(x, y)$. Then

$$f_{\text{prog}} = (\lambda x : A)\text{pr}_1(p) \in (A \to B)$$

is the program extracted from P and

$$f_{\text{corr}} = (\lambda x : A)\text{pr}_2(p) \in (\Pi x : A)P(x, f(x))$$

is the correctness proof for f_{prog}. This in fact provides a general method of extracting programs out of proofs together with proofs of their correctness. But in practice, the algorithms obtained that way are not very efficient, since they contain a large amount of logical meta-information relevant for proof correctness, but not for computation. Thus, it is often desired to rather construct the programs in advance and do the verification afterwards. In this respect, a specification is a functional program mapping A to B and satisfying predicate P is given by the *type* $(\Sigma f : A \to B)P(f)$. So an object of type $(\Sigma f\colon A \to B)P(f)$ is a pair $\langle f, p\rangle$ with $f\colon A \to B$ and p an object in $P(f)$ providing a proof object for the correctness of the functional program f.

The expressiveness of many functional languages is rather restricted, since they lack the implementation of dependent types, which would be required to include constructive logic in the programming language.

Martin-Löf extended type theory in 1970 by adding universes. A universe U is to be thought of as a *big type* of *small types*. It is closed under many type forming operations. This makes it possible to consider types such as $(A : \mathsf{U}) \to A \to A$ or $\mathrm{id} = \lambda A.\lambda x.x \colon (A : \mathsf{U}) \to A \to A$.
If $T(A)$ is a type under the assumption $A : \mathsf{U}$, we can form the dependent type $(A : \mathsf{U}) \to T(A)$. Then M is of this type iff $A : \mathsf{U}$ implies $M(A) : T(A)$. This leads to extensions of type theory whose strength is similiar or even equivalent to that of ZF set theory.
Martin-Löf's introduction of a type of all types was driven by three motivating points:

(i) Bertrand Russell's notion of types as "ranges of significance" of propositional functions

(ii) the need to quantify over all propositions

(iii) the propositions-as-types paradigm

But one can show that the original version of Martin-Löf's theory, where the type of all types contains itself as a term, is inconsistent. The theory contains a non-normalizing proof of $\bot$. This is known as Girard's paradox and it implies that the three points mentioned above cannot be simultaneously satisfied. Martin-Löf decided to take away the second point, leading to a *predicative* type theory.

Models of Intensional Type Theory

In 1973 Martin-Löf introduced a new class of types, denoted by $\mathrm{Id}_A(a, b)$ where A is a type and a, b are terms of type A, the so-called *identity types*. These types can be thought of as the type of proofs that a is equal to b. A distinction has to be made between *definitional equality* on one hand, which means the derivation of equality judgments, and *propositional equality* on the other hand, which is stated by giving a term of an identity type.
An intuitive interpretation of type theory can be given in locally cartesian closed categories, i.e. categories $\mathbf{C}$, whose slice categories $\mathbf{C}/X$ (the categories of objects *over* $X \in \mathrm{ob}(\mathbf{C})$) are all cartesian closed, which is to say, they contain finite products and exponential objects. The latter are used to model dependent function spaces. Modelling type theory on a locally cartesian category gives rise to *extensional* type theory (ETT), where any inhabitant of an identity type already induces a definitional equality, making the distinction between definitional and propositional equality obsolete. But the extensional theory has the major disadvantage of rendering type-checking undecidable. Therefore, intensional type theory is preferable.

In 1994 Thomas Streicher and Martin Hofmann introduced the *groupoid model* for type theory in which a type is interpreted by a *groupoid* and the type $\mathrm{Id}_A(a, b)$ by the set of isomorphisms $a \to b$. There may be more than one inhabitant of an identity type in this model, hence it refutes the so-called principle of *uniqueness of identity proofs*.

Homotopy Type Theory

In 2009 Vladimir Voevodsky introduced a framework for type theory, in which the models are influenced by constructions from homotopy theory. He introduced a hierarchy for types. A type A is called a *proposition* if for all terms a, b of type A, the type $\mathrm{Id}_A(a, b)$ contains precisely one element, i.e. it is *contractible.* Furthermore, A is called a *set* if $\mathrm{Id}_A(a, b)$ is always a proposition and a *groupoid* if $\mathrm{Id}_A(a, b)$ is a set. In fact, any such type can be seen as a groupoid, with the terms as objects, the set of morphisms given by the set $\mathrm{Id}_A(a, b)$. Composition is the proof that equality is transitive, and the identity morphism is the proof that equality is reflexive.
The hierarchy can be extended to higher dimensions, leading to n-groupoids for $n \geq 2$ and ∞-groupoids.

One can introduce a notion of *weak equivalence* between types, which uniformly generalizes the notions of *logical equivalence* between propositions, *bijection* between sets, *categorical equivalence* between groupoids and so on. A map $f\colon A \to B$ is defined to be a *weak equivalence* if for any term b of type B, the type $(\Sigma a : A)\mathrm{Id}_B(f(a), b)$ is a proposition and inhabited. The type of weak equivalences from A to B is written $\mathsf{Weq}(A, B)$. Since $\mathrm{id}_A\colon A \to A$ is a weak equivalence, the type $\mathsf{Weq}(A, A)$ is inhabited for any type A. From this one can conclude the existence of a map

$$\mathrm{Id}_{\mathsf{U}}(A, B) \to \mathsf{Weq}(A, B).$$

Voevodsky's *Univalence Axiom* is equivalent to this map being a weak equivalence. Considering an immediate consequence from it, one could subsume the Univalence Axiom under the slogan "isomorphic types are equal". The Univalence Axiom also implies function extensionality – which cannot be derived in intensional type theory.

Overview

In *Chapter 1* we introduce the basic notions of dependent type theory followed by a comparison of extensional and intensional type theory. We mention the peculiarities of cubical type theory as recently formulated by Coquand et al.

We proceed in *Chapter 2* by giving a model of intensional type theory in the topos of cubical sets whose site is given by finitely generated de Morgan algebras. In contrast to the presentation [CCHM15] by Coquand et al. of the model in type theoretical style, we propound the model in Kripke-Joyal semantics, which is the common framework when modelling logic in a topos. We also state notions of weak equivalences and contractability. The universe will be made out of *fibrant* types, i.e. types endowed with a *composition structure*. Such a structure can be considered a kind of totalization function for paths. Given a partial path in a type, i.e. a path of subterminal objects, if it is connected at 0, it is connected at 1.
Composition structures also allow for defining a variation of the Kan operation for simplicial sets: given an open box, we can add the missing lid. There also exists a derived operation which does the filling.

Also, a *glueing structure* is introduced. Given a weak equivalence from a partial type to a total type, the glueing operation yields a *total* type, extending the former partial type.

This glueing construction plays a central role when defining a composition structure on the universe, which will be done in *Chapter 3*. Assuming a Grothendieck universe on the meta-level, we can lift it to a type-theoretic universe á la Hofmann-Streicher [HS98, Str14c], and also endow it with a composition structure.

Finally, in *Chapter 4* we show that Vladimir Voevodsky's Univalence Axiom is valid in this model.

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1 Dependent Type Theory

1.1 Basic Notions of Dependent Type Theory

Type Theory has originally been inspired by Brouwer's intuitionistic philosophy and Russell's Type Theory. It can be considered a logical framework providing explicit proof objects for which there are rules of computation.
The basic entities of type theory are *types* A and *terms* a of a given type. A *context* Γ is a list of declarations of variables.
The following kinds of judgments can be made:

$\vdash \Gamma$	Γ is a context
$\vdash \Gamma = \Delta$	Γ and Δ are equal contexts
$\Gamma \vdash A$	A is a type in context Γ
$\Gamma \vdash A = B$	A and B are (definitionally) equal types in context Γ
$\Gamma \vdash a : A$	a is a term of type A in context Γ
$\Gamma \vdash a = b : A$	a and b are (definitionally) equal terms of type A in context Γ

In particular, we have *dependent types* which can be thought of as families of types indexed by other types, indicated by a statement of the form

$$x : A \vdash B(x).$$

A statement of the form

$$x = y$$

gets *identified* with the type of proofs of this proposition, the so-called *identity type*

$$\mathrm{Id}_A(x, y),$$

where x and y are *terms* of type A.
The identity type $\mathrm{Id}_A(x, y)$ depends on $x, y : A$. As another example, the type of n-tuples of a type A can be defined by:

$$A^0 := 1$$
$$n : N \vdash A^{n+1} := A \times A^n$$

Here, N denotes the type of natural numbers.
The truth of the arithmetical theorem

$$0 + n = n$$

is witnessed by a proof term

$$e : \mathrm{Id}_N(\mathsf{add}(0,n),n).$$

Besides data types, like e.g. N, $N \to N$ or Bool, also propositions are considered as types of their proof objects.
This paradigm is known as *propositions-as-types* or *Curry-Howard correspondence*. Here, $\forall$-quantifiers are interpreted by "dependent function spaces", so-called Π-types, whilst $\exists$-quantifiers are interpreted by "dependent products", so-called Σ-types.

Proposition	**Type**
$\bot$	0
$\top$	1
$A \vee B$	$A + B$
$A \wedge B$	$A \times B$
$A \implies B$	$A \to B$
$(\exists x : A)B$	$(\Sigma x : A)B(x)$
$(\forall x : A)B$	$(\Pi x : A)B(x)$

In a first attempt, one can interpret these types in a locally cartesian closed category, where $A \times B$ gets interpreted as the product and $A \to B$ the exponential B^A. This provides a model of so-called *extensional type theory (ETT)*.

1.2 Intensional vs. Extensional Type Theory

One has to pay attention when it comes to the derivation of judgments. Having defined addition $\mathsf{add}\colon N \times N \to N$ on the natural numbers N by

$$\mathsf{add}(0,n) = n$$
$$\mathsf{add}(n,\mathsf{succ}(m)) = \mathsf{succ}(\mathsf{add}(n,m))$$

does not in general lead to the derivation of *definitional equality*

$$n : N \vdash \mathsf{add}(0,n) = n,$$

but only to *propositional equality*, which means giving a proof term

$$n : N \vdash e : \mathrm{Id}_N(\mathsf{add}(0,n),n).$$

In extensional type theory, there is no distinction between these two forms of equality, but unfortunately this leads to the drawback of rendering type-checking undecidable.
In contrast, in *intensional type theory (ITT)*, only the latter mentioned statement is derivable.
To prove this, let us assume the reflection rule for identity types

$$\frac{\Gamma \vdash r_A(s){:}\mathrm{Id}_A(s,t)}{\Gamma \vdash s = t : A}\ (\text{Id-R})$$

to hold and simultaneously type-checking to be decidable. Then it would be decidable whether $\vdash s = t : A$ is derivable. Now, for $A = (N \to N)$ and $x = \lambda : N.0$, it would be decidable whether $\vdash \Pi n : N.\mathrm{Id}_N(t(n), 0)$ was provable for closed terms t of type $N \to N$. But this is a contradiction, since there is no recursively enumerable extension T of primitive recursive arithmetic such that the set of Π^0_1-sentences provable in T is decidable.
Hence, in spite of the intuitiveness of the interpretation of ETT in locally cartesian categories, it is better to stick to ITT in many cases.
Identity types are the most distinctive feature in ITT. They are given by the rules

$$\frac{\Gamma \vdash A}{\Gamma, x, y : A \vdash \mathrm{Id}_A(x, y)} \text{(Id-F)}$$

$$\frac{\Gamma \vdash A}{\Gamma, x : A \vdash r_A(x) : \mathrm{Id}_A(x, x)} \text{(Id-I)}$$

$$\frac{\Gamma, x, y{:}A, z{:}\mathrm{Id}_A(x, y) \vdash C(x, y, z) \quad \Gamma, x{:}A \vdash d : C(x, x, r_A(x))}{\Gamma, x, y{:}A, z{:}\mathrm{Id}_A(x, y) \vdash J((x)d)(z) \in C[x, y, z]} \text{(Id-E)}$$

together with the conversion rule

$$J((x)d)(r_A(t)) = d[t/x].$$

In his habilitation thesis [Str93], Thomas Streicher has worked out the following *criteria for intensionality* and has given several models validating them:

(i) $A : \mathsf{Set},\ , x, y : A,\ z : \mathrm{Id}_A(x, y) \not\vdash x = y : A$

(ii) $A : \mathsf{Set},\ B \colon A \to \mathsf{Set},\ \ x, y : A,\ z : \mathrm{Id}_A(x, y) \not\vdash B(x) = B(y) : \mathsf{Set}$

(iii) $\vdash p : \mathrm{Id}_A(t, s) \quad \Longrightarrow \vdash t = s : A$

These models refuted many of those propositions which trivially hold in extensional type theory, such as *function extensionality*

$$(\Pi x : A)\mathrm{Id}_B(f(x), g(x)) \longrightarrow \mathrm{Id}_{A \to B}(f, g)$$

for $A, B : \mathsf{Set}$ and $f, g \colon A \to B$.
These criteria allow for an additional eliminator K for identity types, given by

$$\frac{\Gamma, x{:}A, z{:}\mathrm{Id}_A(x, x) \vdash C(x, z) \quad \Gamma, x{:}A \vdash d : C(x, r_A(x))}{\Gamma, x{:}A, z{:}\mathrm{Id}_A(x, x) \vdash K((x)d)(z) \in C(x, z)} \text{(Id-E')}$$

together with the conversion rule

$$K((x)d)(r_A(t)) = d[t/x].$$

The eliminator K allows one to prove the principle *uniqueness of identity proofs (UIP)*

$$A : \mathsf{Set},\ x, y : A,\ u, v : \mathrm{Id}_A(x, y) \vdash \mathrm{Id}_{\mathrm{Id}_A(x,y)}(u, v).$$

To deal with this issue, Thomas Streicher and Martin Hofmann constructed the *groupoid model* [HS98], where they considered a universe U of small discrete groupoids, such that $A, B : U$ are propositionally equal if and only if they are isomorphic.
During the past years, Vladimir Voevodsky has developed a system called *Homotopy Type Theory (HoTT)*, cf. [Voe09, KLV12, Uni13], which crucially relies on the observations by Streicher and Hofmann.

1.3 Cubical Type Theory

In their recent work [CCHM15], Thierry Coquand et al. provide a model for a type theory extending Martin-Löf type theory by new operations, which allow for a notion of connectedness of types and Kan composition operations. This *cubical* type theory is itself modelled on the category of cubical sets. A cubical set is a presheaf on a site of algebraically defined cubes. In our case, the newly introduced operations crucially rely on the cubes having *connections*, which correspond to minimum and maximum functions on the cubes' coordinates.
This kind of cubical type theory in particular provides a constructive proof of the *univalence axiom*, which roughly states that isomorphic types are equal, corresponding to the common practice in mathematics of identifying isomorphic types. In particular, from the univalence axiom, one can derive function extensionality.

2 Interpreting Type Theory in Cubical Sets

2.1 The Category of Cubical Sets

Definition 2.1.1 (de Morgan Algebra)**.** A *de Morgan algebra* $(A, \wedge, \vee, 0, 1, \neg)$ is a structure such that $(A, \wedge, \vee, 0, 1)$ is a bounded distributive lattice and $\neg \colon A \to A$ is a *de Morgan involution*, i.e. the following laws hold:

$$\neg(x \wedge y) = \neg x \vee \neg y \quad \text{and} \quad \neg\neg x = x$$

Definition 2.1.2 (Cubical Site)**.** We consider the monad (T, μ, η) given by the free de Morgan algebra functor

$$T \colon \mathbf{Set} \to \mathbf{Set},\ I \mapsto T(I) := \mathsf{dM}(I).$$

It turns out that $\mathsf{dM}(I)$ is finite in case I is finite. Let $\mathcal{C}$ denote the opposite category of the Kleisli category of this monad, when restricted to $\mathbf{Set}_{\text{fin}}$, i.e.

$$\begin{aligned} \text{ob}(\mathcal{C}) &= \text{ob}(\mathbf{Set}_{\text{fin}}), \\ \text{Hom}_{\mathcal{C}}(J, I) &= \text{Hom}_{\mathbf{Set}}(I, \mathsf{dM}(J)). \end{aligned}$$

Accordingly, the identity morphisms in $\mathcal{C}$ are given by the units

$$\eta_I \colon I \to \mathsf{dM}(I),$$

and composition of morphisms $f \in \text{Hom}_{\mathcal{C}}(J, I)$, $g \in \text{Hom}_{\mathcal{C}}(K, J)$ is given by the composite

$$I \xrightarrow{f} \mathsf{dM}(J) \xrightarrow{\mathsf{dM}(g)} \mathsf{dM}(\mathsf{dM}(K)) \xrightarrow{\mu_K} \mathsf{dM}(K).$$

The category $\mathcal{C}$ is called the *category of cubes*.
In case $i \notin I$, we write $I, i := I + \{i\}$ for the disjoint union. If $i \in I$, we write $I - i := I \setminus \{i\}$ for the set-theoretic complement.

We remark that $\mathcal{C}$ has finite products, since the category of de Morgan algebras has finite sums. In fact

$$\mathsf{dM}(I) \times \mathsf{dM}(J) \cong \mathsf{dM}(I + J)$$

for $I, J \in \text{ob}(\mathcal{C})$.

Definition 2.1.3 (Face Maps and Strict Maps)**.** A *face map* is a composition of maps of the form $(ib) \in \mathrm{Hom}_{\mathcal{C}}(I - i, I)$, where

$$(ib)(x) := \begin{cases} x & \text{if } x \neq i \\ b & \text{if } x = i \end{cases}$$

for $b = 0, 1$.
A *strict map* in $\mathcal{C}$ is a map which contains neither 0, nor 1 in its image.

Theorem 2.1.4. *Any map f in $\mathcal{C}$ can be written as a composition $f = gh$ where g is a face map and h is a strict map.*

Definition 2.1.5 (The Category of Cubical Sets)**.** The presheaf category $\mathbf{cSet} := \widehat{\mathcal{C}} = \mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$ is called the *category of cubical sets.*
A *cubical set* is a presheaf $\mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$.

Definition 2.1.6 (Interval Object)**.** The *interval object* in **cSet** is given by

$$\mathbb{I} = \mathrm{Hom}_{\mathcal{C}}(-, \{\emptyset\}) = \mathrm{Hom}_{\mathcal{C}}(-, 1).$$

There are morphisms

$$1 \underset{0}{\overset{1}{\rightrightarrows}} \mathbb{I}$$

defined by

$$0_I(!_I)(*) = 0 = \min(\mathsf{dM}(I)) \quad \text{and} \quad 1_I(!_I)(*) = 1 = \max(\mathsf{dM}(I)).$$

For $f \colon J \to I$, the induced morphism is postcomposition with f, i.e. $\mathbb{I}(f) = f \circ -$.

We remark that for $f \colon J \to I$ in $\mathcal{C}$, we have

$$\mathbb{I}(I) = \mathrm{Hom}_{\mathcal{C}}(I, 1) = \mathrm{Hom}_{\mathbf{Set}}(1, \mathsf{dM}(I)) \cong \mathsf{dM}(I)$$

and $\mathbb{I}(f) \colon \mathbb{I}(I) \to \mathbb{I}(J)$ is a morphism of de Morgan algebras. In particular, $\mathbb{I}$ has *connections*

$$\mathbb{I} \times \mathbb{I} \underset{\wedge}{\overset{\vee}{\rightrightarrows}} \mathbb{I}$$

given by the infimum and supremum, resp. Also, there is *inversion* given by the involution

$$\mathbb{I} \xrightarrow{\neg} \mathbb{I}.$$

As a category of **Set**-valued presheaves, $\mathbb{E} = \mathbf{cSet}$ is a topos, as is well-known. Therefore it has a subobject classifier Ω, where

$$\Omega(I) := \mathrm{Sub}(\mathbf{y}_{\widehat{\mathcal{C}}}(I)),$$

which is in bijection with the set of sieves on I for $I \in \mathrm{ob}(\mathcal{C})$.
Let $(\cdot = 1) \colon \mathbb{I} \to \Omega$ be the presheaf morphism defined by

$$(\cdot = 1)_I(r) := (r = 1)_I := \{\ f \colon J \to I \quad | \quad (\mathbb{I}(f)(r))(*) = 1\} \in \mathrm{Sv}(I).$$

Definition 2.1.7 (Face Lattice)**.** The *face lattice* $\mathbb{F}$ is the image presheaf of this morphism, i.e.:

$$\begin{array}{ccc} \mathbb{I} & \xrightarrow{(\cdot=1)} & \Omega \\ & \searrow \quad \nearrow & \\ & \mathbb{F} := \mathrm{im}(\cdot = 1) & \end{array}$$

By restriction, $\mathbb{F}$ is an internal lattice, as follows:

$$\begin{array}{ccc} \mathbb{F} \times \mathbb{F} & \overset{\wedge}{\dashrightarrow} & \mathbb{F} \\ {\scriptstyle i \times i}\downarrow & & \downarrow{\scriptstyle i} \\ \Omega \times \Omega & \underset{\wedge}{\longrightarrow} & \Omega \end{array}$$

In fact, the face lattice can be represented as the free distributive lattice on $i, 1-i$ for all $i \in I$.
There are quotient maps $\mathbb{I}(I) \twoheadrightarrow \mathbb{F}(I)$. Since for each $x \in \mathbb{I}(I)$ the element $x \wedge (1-x)$ gets mapped to 0 under the respective quotient map, the induced map $\mathbb{F}(f)$ in

$$\begin{array}{ccc} \mathbb{I}(I) & \xrightarrow{\mathbb{I}(f)} & \mathbb{I}(J) \\ \downarrow & & \downarrow \\ \mathbb{F}(I) & \underset{\mathbb{F}(f)}{\dashrightarrow} & \mathbb{F}(J) \end{array}$$

is well-defined.

2.2 Interpreting Type Theory in Cubical Sets

2.2.1 Contexts and Types

Contexts are interpreted as cubical sets $\Gamma \in \mathbf{cSet}$. We consider the *category of elements* $\mathbf{Elts}_{\mathcal{C}}\Gamma$ whose objects are pairs (I, ρ), where $I \in \mathrm{ob}(\mathcal{C})$, $\rho \in \Gamma(I)$. A morphism $(I,\rho) \to (J,\tau)$ is given by a morphism $f \colon J \to I$ in $\mathcal{C}$ with $\Gamma(f)(\rho) = \tau$.
Furthermore, one can identify $\mathbf{Elts}_{\mathcal{C}}\Gamma = \mathbf{y}_{\mathcal{C}} \downarrow \Gamma$ by the Yoneda Lemma.
It is well-known, that there exists an equivalence

$$\widehat{\mathbf{Elts}_{\mathcal{C}}\Gamma} \simeq \widehat{\mathcal{C}} \downarrow \Gamma.$$

Types over Γ are interpreted as presheaves $A \in \mathsf{Ty}(\Gamma) := \widehat{\mathbf{Elts}_{\mathcal{C}}\Gamma}$. Hence, a type A is given by a family of sets $(A(I,\rho))_{I \in \mathcal{C}, \rho \in \Gamma(I)}$ with restriction maps

$$Af \colon A(I,\rho) \to A(J, \Gamma(f)(\rho))$$

for $f \colon J \to I$ in $\mathcal{C}$.

2.2.2 Context Comprehension and Terms

Given a type $A \in \mathsf{Ty}(\Gamma)$ we interpret context comprehension $p_A^\Gamma \colon \Gamma.A \to \Gamma$ by the presheaf

$$(\Gamma.A)(I) := \{\langle \rho, u\rangle \colon \rho \in \Gamma(I),\ u \in A(I, \rho)\}$$

with restrictions

$$(\Gamma.A)(f) \colon (\Gamma.A)(I) \to (\Gamma.A)(J),\ \langle \rho, u\rangle \mapsto \langle \Gamma(f)(\rho), A(f)(u)\rangle$$

and the presheaf morphism $p_A := p_A^\Gamma \colon \Gamma.A \to \Gamma$ given by first projection

$$(p_A)_I \colon (\Gamma.A)(I) \to \Gamma(I),\ \langle \rho, u\rangle \mapsto \rho.$$

Terms of type A are interpreted as sections of the morphisms $p_A \colon \Gamma.A \to \Gamma$, i.e. maps $a \colon \Gamma \to \Gamma.A$ such that $p_A \circ a = \mathrm{id}_\Gamma$.
For $I \in \mathrm{ob}(\mathcal{C})$ and $\rho \in \Gamma(I)$ this means $(p_A)_I(a_I(\rho)) = \rho$.
Hence, we can write

$$a_I(\rho) = \langle \rho, a_\rho\rangle \in (\Gamma.A)(I).$$

We denote the set of terms of type $A \in \mathsf{Ty}(\Gamma)$ by

$$\mathsf{Ter}(\Gamma; A) := \{a \colon \Gamma \to \Gamma.A \ \mid\ p_A \circ a = \mathrm{id}_\Gamma\}.$$

2.2.3 Context Substitution

Given contexts Δ and Γ, substitution is modelled by morphisms $\sigma \colon \Delta \to \Gamma$.
In **cSet**, we have a functorial choice of pullbacks

$$\begin{array}{ccc} \Delta.(\sigma^* A) & \xrightarrow{q} & \Gamma.A \\ \big\downarrow & \lrcorner & \big\downarrow \\ \Delta & \xrightarrow[\sigma]{} & \Gamma \end{array}$$

so σ induces a functor $\sigma^* \colon \widehat{\mathcal{C}}/\Gamma \to \widehat{\mathcal{C}}/\Delta$, assigning to a type $A \in \mathsf{Ty}(\Gamma)$ the base change $\sigma^* A \in \mathsf{Ty}(\Delta)$.
For $I \in \mathrm{ob}(\mathcal{C})$ and $\delta \in \Delta(I)$ we have

$$(\sigma^* A)(I, \delta) = A(I, \sigma(\delta)).$$

A term $a \in \mathsf{Ter}(\Gamma; A)$ yields a term $\sigma^* a \in \mathsf{Ter}(\Delta; \sigma^* A)$ by

$$(\sigma^* a)_I(\delta) := \langle \delta, a_{\sigma(\delta)}\rangle,$$

where $a_I(\rho) = \langle \rho, a_\rho\rangle \in (\Gamma.A)(I)$ for $\rho \in \Gamma(I)$.

2.2.4 Dependent Sums and Products

Definition 2.2.1 (Dependent Sum)**.** Let $A \in \mathsf{Ty}(\Gamma)$, $B \in \mathsf{Ty}(\Gamma.A)$. The *dependent sum of* A *and* B is the type $\Sigma_\Gamma(A, B) \in \mathsf{Ty}(\Gamma)$ defined by

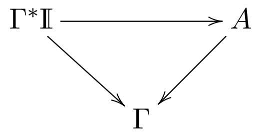
$$\Sigma_\Gamma(A, B)(I, \rho) := \{\langle u, v\rangle : u \in A(I, \rho),\ v \in B(I, \rho, v)\}.$$

Definition 2.2.2 (Dependent Product)**.** Let $A \in \mathsf{Ty}(\Gamma)$, $B \in \mathsf{Ty}(\Gamma.A)$. For the base change functor $(p_A)^* : \widehat{\mathcal{C}}/\Gamma \to \widehat{\mathcal{C}}/\Gamma.A$, consider the right adjoint $\Pi_{p_A} : \widehat{\mathcal{C}}/\Gamma.A \to \widehat{\mathcal{C}}/\Gamma$, which exists due to [MM92, Sect. IV,7; Thm. 2].
We define the *dependent product of* A *and* B, written $\Pi_\Gamma(A, B)$, by

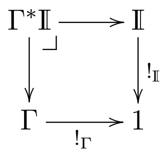
$$p_{\Pi_\Gamma(A,B)} := \Pi_{p_A}(p_B) : \Gamma.\Pi_\Gamma(A, B) \to \Gamma.$$

2.2.5 Identity and Path Types

Let $A \in \mathsf{Ty}(\Gamma)$. The exponential object $A^{\mathbb{I}}$ in the slice category $\widehat{\mathcal{C}}/\Gamma$ is given by

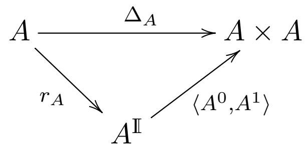


where:



With this at hand, we can define the identity type as an object of $\widehat{\mathcal{C}}/\Gamma$.

Definition 2.2.3 (Identity Type)**.** The *identity type of* A is the factorisation of the diagonal map $\Delta_A = \langle \mathrm{id}_A, \mathrm{id}_A\rangle : A \to A \times A$ as in:



Definition 2.2.4 (Path Type)**.** Let $A \in \mathsf{Ty}(\Gamma)$ and $u, v : \Gamma \to A$ be terms of type A. For $i \notin I$, the map $s_i : (I, i) \rightarrowtail I$ is induced by inclusion. For $\Gamma \in \mathbf{cSet}$, we can consider the type $\Gamma'(I, \rho) := \Gamma(I)$, which we also denote by Γ.
We define the *type of paths from* u *to* v *in* A by

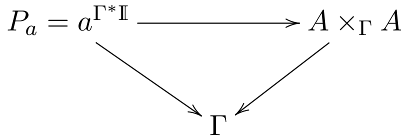
$$\begin{aligned} P_A(u, v)(I, \rho) := \{[w]_\sim : w \in A((I, i), \Gamma(s_i)(\rho)), \\ \text{s.t.: } u_{(I, \Gamma(s_i \circ (i0))(\rho))}(\rho) = A(i0)(w),\ \ v_{(I, \Gamma(s_i \circ (i1))(\rho))}(\rho) = A(i1)(w)\}, \end{aligned}$$

where the equivalence relation $\sim$ is defined by

$$w' = A(i/j)(w)$$

for $(i/j)\colon I, j \to I, i$ arising from the substitution $i \mapsto j$.

For a term $a\colon \Gamma \to A$, in fact $P_A(a,a) = P_a$ corresponds to the exponential in the slice category, given by:



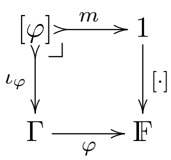
2.3 Types with Composition Structure

2.3.1 Composition Structures and Fibrant Types

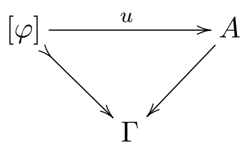
Definition 2.3.1 (Partial Elements and Partial Paths)**.** Let $\varphi\colon \Gamma \to \mathbb{F}$ be a morphism. For a cubical set Γ, we consider the cubical set $(\Gamma, \varphi) := [\varphi]$ defined by

$$(\Gamma, \varphi)(I) := \{\rho \in \Gamma(I)\colon \varphi_I(\rho) = 1_{\mathbb{F}}\}.$$

A *partial element of extent* $[\varphi]$ of Γ is a monomorphism $m\colon [\varphi] \rightarrowtail 1$, i.e. we have the following diagram



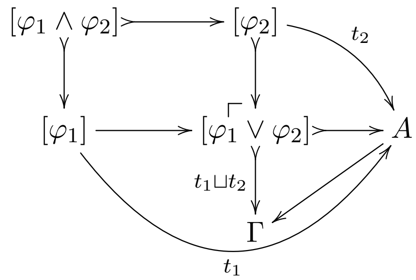
where $[\cdot]\colon 1 \to \mathbb{F}$ picks the largest element in every fiber $\mathbb{F}(I)$.
Let A be a type over Γ. A *partial path of extent* $[\varphi]$ in A is a map $u\colon [\varphi] \to A$ such that



commutes.

A morphism $\varphi\colon \Gamma \to \mathbb{F}$ can be seen as a predicate on Γ. The *join* of partial elements then corresponds to disjunction.

Definition 2.3.2 (Join of Partial Elements and Partial Paths)**.** Given $\iota_i \colon [\varphi_i] \to A$, $i = 1, 2$, we can form the pushout (cf. [MM92, p. 186; Proof 3]):

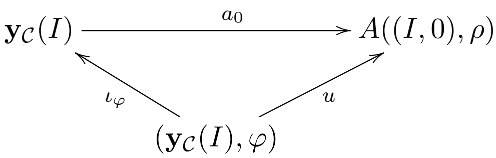


Then $[\varphi_1 \vee \varphi_2]$ is called the *join* of the partial elements φ_1 and φ_2 (cf. [OP16, Def. 4.2], and $t_1 \sqcup t_2$ the *join* of the partial paths t_1 and t_2, resp.

Definition 2.3.3 (Composition Structures and Fibrant Types)**.** Let $A \in \mathsf{Ty}(\Gamma)$. A *composition structure on* A is a family of maps, given for all (I, i), with $I \in \mathrm{ob}(\mathcal{C})$, $i \notin I$, $\rho \in \Gamma(I, i)$, $\varphi \in \mathbb{F}(I)$ and $u \in \mathsf{Ter}((\Gamma, \varphi); A\iota_\varphi)$, by

$$\mathsf{comp}_\Gamma(A, (I, i), \rho, \varphi, u, -) \colon A((I, 0), \rho)|_{\mathrm{ext}(u)} \to A((I, 1), \rho), \ a_0 \mapsto a_1,$$

where $\mathrm{ext}(u)$ is the subset of elements $a_0 \in A((I, 0), \rho)$ extending the partial element u:

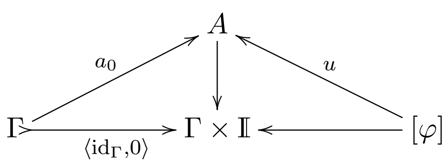


We require the operation to be *uniform* in the following sense: For any morphism $f \colon (I, i) \to (J, j)$, $j \notin J$, $f(i) = j$, it holds that

$$\mathsf{comp}_\Gamma\big((I, i), \rho, \varphi, u, a_0\big) = Af\left(\mathsf{comp}_\Gamma\left((J, j), \rho \circ \mathbf{y}_\mathcal{C}(f), \varphi \circ \mathbf{y}_\mathcal{C}(f)|_{(\mathbf{y}_\mathcal{C}(J,j),\varphi)}, a_0 \circ \mathbf{y}_\mathcal{C}(f)\right)\right).$$

A *fibrant type* is a type together with a composition structure $(A, \mathsf{comp}_\Gamma)$. We write $\mathsf{FTy}(\Gamma)$ for all fibrant types over Γ.

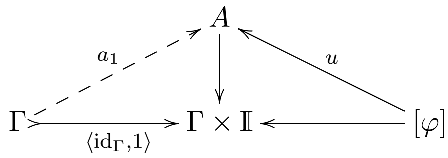
We lift this operation to $\widehat{\mathbf{Elts}_\mathcal{C}\Gamma} \simeq \widehat{\mathcal{C}}/\Gamma$. Correspondingly, a composition structure on $A \in \widehat{\mathcal{C}}/\Gamma$ is a family of maps, assigning to each $\varphi \colon \Gamma \to \mathbb{F}$, partial path $u \colon [\varphi] \to A$, and term $a_0 \colon \Gamma \to A$ as in



an element

$$\mathsf{comp}_\Gamma(A, \varphi, u, a_0) =: a_1$$

as in:



The operation is *uniform* in the following sense: For any substitution $\sigma\colon \Delta \to \Gamma$ it holds that:

$$\alpha_\sigma \circ \underbrace{\mathsf{comp}_\Delta(\sigma^* A, \varphi \circ \sigma, (\alpha_\sigma)^* u, a_0 \circ \sigma)}_{=:\sigma^*\mathsf{comp}_\Gamma(A,\varphi,u,a_0)} = \mathsf{comp}_\Gamma(A, \varphi, u, a_0) \circ \sigma.$$

2.3.2 Operations derived from Composition

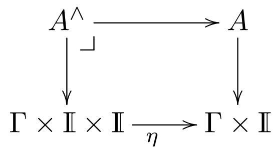
Kan Filling

The composition operation allows for defining a filling operation which provides a notion of the classically well-known Kan complexes for cubical sets. Note that this yields the Kan fillings as additional *structure*, rather than a mere property.

Definition 2.3.4 (Kan Filling Operation)**.** Let $A \in \mathsf{Ty}(\Gamma \times \mathbb{I})$. For a partial path u of extent φ, $a_0\colon \Gamma \to A$, we define a variant of the classically known *Kan filling*, which is *uniform* (cf. [OP16, GS15, CCHM15]):

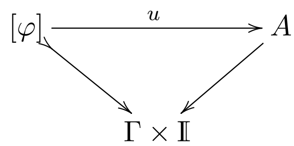
$$\mathsf{fill}_\Gamma(A, \varphi, u, a_0) := \mathsf{comp}_{\Gamma\times\mathbb{I}}(A^\wedge, \varphi', (\alpha_\eta)^* u, a_0'),$$

where



with $\eta := \Gamma \times \wedge\colon \Gamma\times\mathbb{I}\times\mathbb{I} \to \Gamma\times\mathbb{I}$, $\varphi' := \varphi \circ \mathrm{pr}_1\colon \Gamma\times\mathbb{I} \to \mathbb{F}$ and $a_0' := a_0 \circ \mathrm{pr}_1\colon \Gamma\times\mathbb{I} \to A$.

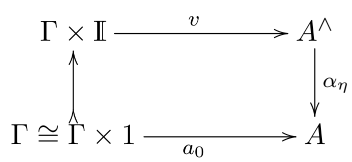
Thus, given a partial element as in

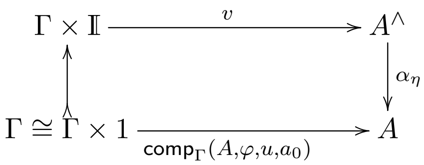


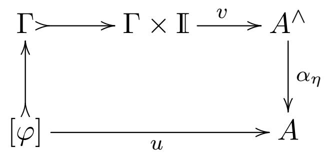
and setting

$$v := \mathsf{fill}_\Gamma(A, \varphi, u, a_0)$$

yields an element $v\colon \Gamma \times \mathbb{I} \to A^\wedge$ such that the following diagrams commute:







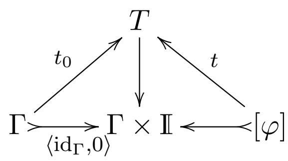
Transport Operation

Definition 2.3.5 (Transport Operation)**.** In the case $\varphi = 0_{\mathbb{F}}$, we have that $[\varphi] = \emptyset$ is the initial cubical set. For $A \in \mathsf{Ty}(\Gamma \times \mathbb{I})$ and $a\colon \Gamma \to A$, we define the *transport operation*

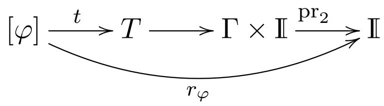
$$\mathsf{transp}(A, a) := \mathsf{comp}_{\Gamma\times\mathbb{I}}(A, 0_{\mathbb{F}}, 0, a).$$

Preservation Operation

Filling allows for an operation witnessing that composition of paths is preserved by maps. Let $A, T \in \mathsf{Ty}(\Gamma \times \mathbb{I})$. Assume, we are given a map $f\colon \mathbb{I} \to A^T$ (which can be viewed as the transpose of a homotopy $\widehat{f}\colon T \times \mathbb{I} \to A$) and a partial path $t\colon [\varphi] \to T$ as in:



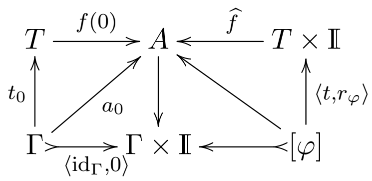
Now, for the composite



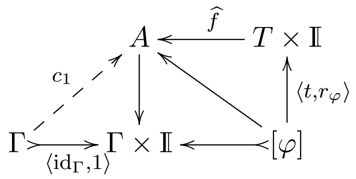
we set

$$c_1 := \mathsf{comp}_\Gamma(A, \varphi, \widehat{f} \circ \langle t, r_\varphi\rangle, f(0) \circ t_0), \quad c_2' := \mathsf{comp}_\Gamma(T, \varphi, t, t_0) \quad \text{and} \quad c_2 := f(1)(c_2').$$

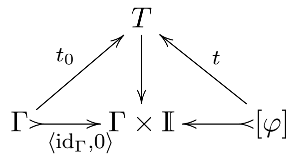
Thus, defining $a_0 := f(0) \circ t_0$, from



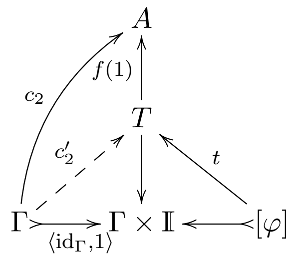
we obtain a totalization



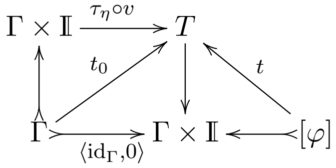
and analogously the situation



yields:



We set $v := \mathsf{fill}_\Gamma(T, \varphi, t, t_0)$ so as to obtain a diagram



for $\tau_\eta \colon T^\wedge = \eta^* T \to T$.
We are now able to define the operation pres:

Definition 2.3.6 (Preservation Operation)**.** Let $f : \mathbb{I} \times T \to A$, a partial element $t_0 \colon \Gamma \to T$ and a partial path $t \colon [\varphi] \to T$. We define the following operation:

$$\mathsf{pres}(f, \varphi, t, t_0) := (\lambda j : \mathbb{I})\mathsf{comp}_\Gamma(A, \varphi \vee (j = 1), f \circ \langle v, r_\varphi \rangle, a_0)$$

Theorem 2.3.7. *The term* $\mathsf{pres}(f, \varphi, t, t_0)$ *is of type* $P_{A(1)}(c_1, c_2)$.

Proof. Let $p := \mathsf{pres}(f, \varphi, t, t_0)$. By definition of the filling operation, we have $v(0) = t_0$, so

$$p(0) = \mathsf{comp}_\Gamma(A, \varphi, f \circ \langle t_0, r_\varphi \rangle, a_0) = c_1.$$

Furthermore, due to $v(1) = c_2'$ and the uniformity of composition, we obtain

$$p(1) = \mathsf{comp}_\Gamma(A, \varphi, f \circ \langle \mathsf{comp}_\Gamma(T, \varphi, t, t_0), r_\varphi \rangle, a_0) = f(1)(\mathsf{comp}_\Gamma(T, \varphi, t, t_0)) = c_2.$$

Now, since $v(j) = t$ is of type $T(1)$ over (Γ, φ), we have that $p(j)$ is of type $A(1)$. $\square$

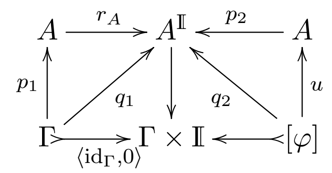
2.3.3 Contractible Types and Weak Equivalences

Contractability

To express contractability of a type A, we define:

$$\mathsf{isContr}(A) := (\Sigma x : A)(\Pi y : A)P_A(x, y)$$

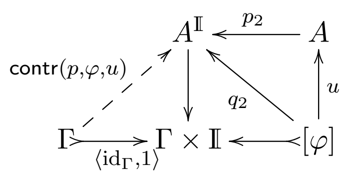
We consider a term p of type $\mathsf{isContr}(A)$ and a partial path $u\colon [\varphi] \to A$. For $i = 1, 2$, let $p_i := \mathrm{pr}_i \circ p$. Then we have a term $p_1\colon \Gamma \to A$ and a function $p_2\colon A \to A^{\mathbb{I}}$. Setting $q_1 := r_A \circ p_1$ and $q_2 := p_2 \circ u$ yields the following situation:



Now, we define the operation

$$\mathsf{contr}(p, \varphi, u) := \mathsf{comp}_\Gamma(A^{\mathbb{I}}, \varphi, q_2, q_1),$$

which yields:



Weak Equivalences

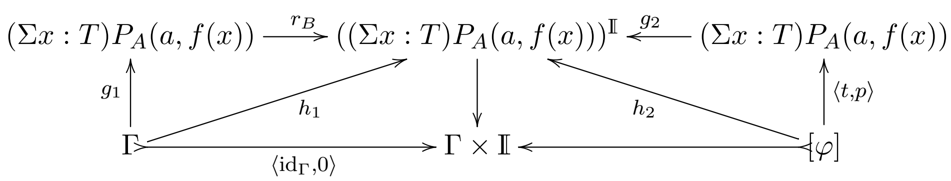
For a map $f\colon T \to A$ to be a weak equivalence is expressed by:

$$\mathsf{isWeq}(T, A, f) := (\Pi y : A)\mathsf{isContr}((\Sigma x : T)P_A(y, f(x)))$$

So the space of weak equivalences is taken to be:

$$\mathsf{Weq}(T, A) := (\Sigma(f\colon T \to A))\mathsf{isWeq}(T, A, f)$$

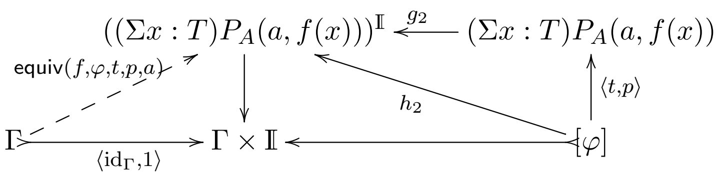
Assume we are given $a\colon \Gamma \to A$, a partial element $t\colon [\varphi] \to T$, a term p of type $P_A(a, f(t))$ and a weak equivalence, i.e. a term f of type $\mathsf{Weq}(T, A)$.
For $B := (\Sigma x : T)P_A(y, f(x))$, we have $\mathrm{pr}_2 \circ f := f_2 \in (\Pi y : A)\mathsf{isContr}(B)$ and $f_2(a) \in \mathsf{isContr}((\Sigma x : T)P_A(a, f(x)))$. Writing $g_1 := \mathrm{pr}_1(f_2(a))\colon \Gamma \to B$, $g_2 := \mathrm{pr}_2(f_2(a))\colon B \to B^{\mathbb{I}}$, $h_1 := r_B \circ g_1$ and $h_2 := g_2 \circ \langle t, p\rangle$, we have:



Hence, we can define the operation

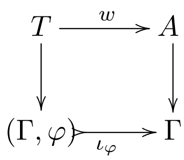
$$\mathsf{equiv}(f, \varphi, t, p, a) := \mathsf{contr}((\mathrm{pr}_2 \circ f)(a), \varphi, \langle t, p \rangle),$$

which gives:



2.3.4 The Glueing Operation

Let Γ be a context and $\varphi \colon \Gamma \to \mathbb{F}$. Assume a type A over Γ, a *partial* type T of extent φ and a weak equivalence $w \colon T \to A$, i.e.:



We want to define an operation, that turns T into a total type $B = \mathsf{Glue}_\Gamma(\varphi, T, A, w)$ and yields a map $\mathsf{unglue} \colon B \to A$ such that B and unglue are extensions of T and f, resp.

Definition 2.3.8 (Glueing Operation)**.** For $\varphi \colon \Gamma \to \mathbb{F}$, $T \in \mathsf{Ty}(\Gamma, \varphi)$, $A \in \mathsf{Ty}(\Gamma)$ and a partial element w of $\mathsf{Weq}(T, \iota_\varphi^* A) \in \mathsf{Ty}(\Gamma, \varphi)$, we define the *glueing operation* as the *total* type $\mathsf{Glue}_\Gamma(\varphi, T, A, w) \in \mathsf{Ty}(\Gamma)$ where:
For $I \in \mathrm{ob}(\mathcal{C})$ and $\rho \in (\Gamma, \varphi)(I)$, let

$$\mathsf{Glue}_\Gamma(\varphi, T, A, w)(I, \rho) := \begin{cases} T(I, \rho) & \text{if } \varphi = 1_{\mathbb{F}} \\ \{\langle t, a \rangle \colon t \in T(I, \rho), a \in A(I, \rho),\ w_{(I,\rho)}(t) = a\} & \text{else} \end{cases}$$

and for $f \colon J \to I$, let

$$\mathsf{Glue}_\Gamma(\varphi, T, A, w)f := \begin{cases} Tf & \text{if } \varphi = 1_{\mathbb{F}} \\ \mathsf{glue}(\varphi, t, a)f & \text{else} \end{cases}$$

where:

$$\mathsf{glue}(\varphi, t, a)f := \begin{cases} Tf(t) & \text{if } \varphi_\rho(f) = 1_{\mathbb{F}} \\ \{\langle Tf(t), Af(a) \rangle \colon t \in T(I, \rho), a \in A(I, \rho)\} & \text{else} \end{cases}$$

Hence, glueing can be seen as a witness for the connectedness of the partial element φ.

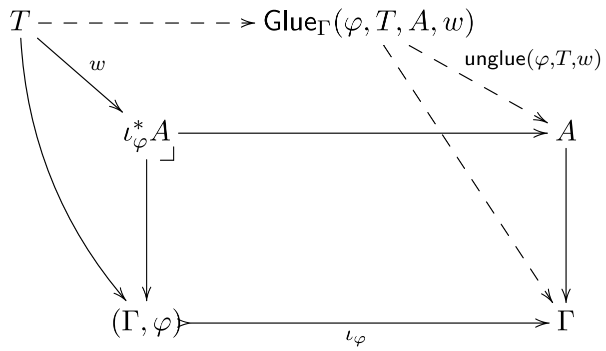
Furthermore, we consider

$$\mathsf{unglue}(\varphi, T, w) \colon \Gamma.\mathsf{Glue}_\Gamma(\varphi, T, A, w) \to A$$

defined by:

$$\mathsf{unglue}(\varphi, T, w)(\rho, \mathsf{glue}(\varphi, t, a)) := \begin{cases} w_{(I,\rho)}(t) & \text{if } \varphi_I(\rho) = 1_{\mathbb{F}} \\ a & \text{else} \end{cases}$$

The glueing operation is depicted in the diagram:



In particular, this means

$$\mathsf{Glue}_\Gamma(1_{\mathbb{F}}, T, A, w) = T$$

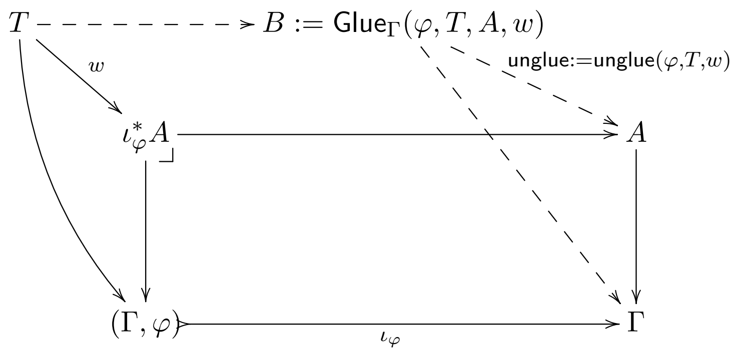
and

$$\mathsf{Glue}_\Gamma(\varphi, T, A, w)\sigma = \mathsf{Glue}_\Delta(\varphi \circ \sigma, T\sigma, A\sigma, \sigma^* w)$$

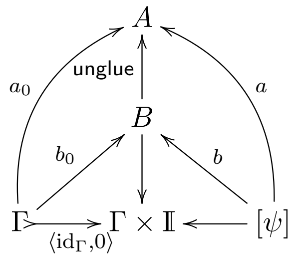
for a change of base $\sigma \colon \Delta \to \Gamma$.

2.3.5 Composition for Glueing

Assume the situation



with



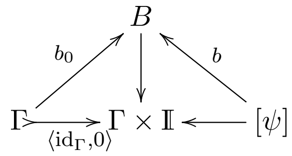
where $a := \mathsf{unglue} \circ b$ and $a_0 := \mathsf{unglue} \circ b_0$.

Theorem 2.3.9. *There is a composition structure on* $B = \mathsf{Glue}_\Gamma(\varphi, T, A, w) \in \mathsf{Ty}(\Gamma)$.

Proof. This is due to [CCHM15, Sec. 6.2]. For $i \in I$, we define $\forall i.\varphi \colon \Gamma \to \mathbb{F}$ as the disjunction

$$\forall i.\varphi = \bigvee_{\substack{\psi \leq \varphi \\ \psi \text{ indep. of } i}} \psi.$$

Let $B := \mathsf{Glue}_\Gamma(\varphi, T, A, f)$. Let $\psi \colon \Gamma \to \mathbb{F}$ and b be a partial element of B at extent $[\psi]$ with:



For $a = \mathsf{unglue} \circ b$ and $a_0 = \mathsf{unglue} \circ b_0$, let

$$a_1' \colon \Gamma \to A, \quad a_1' := \mathsf{comp}_\Gamma(A, \psi, a, a_0),$$

and

$$t_1' \colon (\Gamma, \delta) \to B, \quad t_1' := \mathsf{comp}_\Gamma(T, \psi, b, b_0).$$

By means of the equiv-operation, we can then construct a path α in $P_A(a_1', f(t_1'))$ over the restricted context $(\Gamma, \varphi(1))$.
We now let

$$a_1 := \mathsf{comp}_\Gamma(A(1), \varphi(1) \vee \psi, \alpha \sqcup a(1), a_1').$$

and

$$b_1 := \mathsf{glue}(\varphi(1), t_1, a_1).$$

Then, on the extent $[\delta]$, we have $B = T$ as well as

$$b_1 = \mathsf{comp}_{(\Gamma,\delta)}(T, \psi, b, b_0).$$

□

3 Interpreting Universes in Cubical Sets

3.1 Universes in Toposes

Definition 3.1.1 (Universe in a Topos)**.** Let $\mathbb{E}$ be a topos. A *universe in* $\mathbb{E}$ is a class $\mathbf{U}$ of morphisms in $\mathbb{E}$ such that:

(i) $\mathbf{U}$ is stable under pullbacks along morphisms in $\mathbb{E}$: if $a\colon A \to I$ in $\mathbf{U}$, and $f\colon J \to I$ is a morphism in $\mathbf{U}$, then $f^*a\colon B \to J$ is in $\mathbf{U}$

(ii) $\mathbf{U}$ contains all monomorphisms of $\mathbb{E}$

(iii) $\mathbf{U}$ is closed under dependent sums:[1] if $f\colon A \to I$ and $g\colon B \to A$ are in $\mathbf{U}$, then $\Sigma_f(g) := f \circ g \in \mathbf{U}$.

(iv) $\mathbf{U}$ is closed under dependent products: if $f\colon A \to I$ and $g\colon B \to A$ are in $\mathbf{U}$, then $\prod_f(g) \in \mathbf{U}$.

(v) In $\mathbf{U}$ there is a *generic morphism*, i.e. a morphism $\mathrm{el}\colon E \to U$ such that for each $a\colon A \to I$ in $\mathbf{U}$, there is a morphism $f\colon I \to U$ such that:

$$\begin{array}{ccc} A & \longrightarrow & E \\ {\scriptstyle a}\downarrow & \lrcorner & \downarrow{\scriptstyle \mathrm{el}} \\ I & \xrightarrow[f]{} & U \end{array}$$

This means $a \cong f^*\mathrm{el}$. In this sense, $\mathrm{el}\colon E \to U$ *weakly* classifies the morphisms in $\mathbf{U}$, i.e. the base change morphism f is not required to be unique.

A universe $\mathbf{U}$ in $\mathbb{E}$ is called *impredicative* if $\Omega \to 1$ is in $\mathbf{U}$.

3.2 A Universe in Cubical Sets

Let $\mathcal{U}$ be a universe in the ambient set theory. The following construction is due to [HS98, Str14b, KLV12].
We consider

$$\mathsf{Ty}_0(\Gamma) := \{A \in \mathsf{Ty}(\Gamma)\colon A(I,\rho) \in \mathcal{U} \text{ f.a. } I \in \mathrm{ob}(\mathcal{C}),\ \rho \in \Gamma(I)\}$$

[1]Recall that there is a chain of adjunctions $\Sigma_f \dashv f^* \dashv \Pi_f \colon \mathbb{E}/I \to \mathbb{E}/A$, where $f^*\colon \mathbb{E}/A \to \mathbb{E}/I$ denotes pullback along f.

and

$$\mathsf{FTy}_0(\Gamma) := \{\langle A, \mathsf{comp}_\Gamma\rangle \colon A \in \mathsf{Ty}_0(\Gamma)\}.$$

Then Ty_0 and FTy_0 are sub-presheaves of Ty and FTy, resp.
We consider the cubical sets

$$\mathsf{U}(I) := \mathsf{FTy}_0(\mathbf{y}_\mathcal{C}(I)), \quad \mathsf{U}(f) := \Sigma_{\mathbf{y}_\mathcal{C}(f)} = \mathbf{y}_\mathcal{C}(f) \circ -$$

and

$$\tilde{\mathsf{U}}(I) := \{\langle A, a\rangle \ : A \in \mathsf{U}(I), \quad a \in A(\mathrm{id}_I)\},$$
$$\tilde{\mathsf{U}}(f)(\langle A, a\rangle) := \langle \mathsf{U}(f)(A), A(f \xrightarrow{f} \mathrm{id}_I)(a)\rangle.$$

By $p_\mathsf{U} \colon \tilde{\mathsf{U}} \to \mathsf{U}$ we denote the first projection.
Now $\mathsf{U}(I)$ is a subcategory of $\mathcal{U}^{(\mathcal{C}/I)^{\mathrm{op}}}$, which in turn is equivalent to the subcategory of $\widehat{\mathcal{C}}/\mathbf{y}_\mathcal{C}(I)$ of morphisms with $\mathcal{U}$-small fibers and can be considered as family of presheaves over $\mathbf{y}_\mathcal{C}(I)$.

3.3 Composition for the Universe

We show that the universe just defined carries a composition structure.
For $A, B \in \mathsf{Ty}(\Gamma)$, let $E \in \mathsf{Ty}(\Gamma \times \mathbb{I})$ in $P_\mathsf{U}(A, B)$ be a path in the universe.
We claim that the type $\mathsf{Weq}(A, B)$ contains a canonical inhabitant.

Theorem 3.3.1. *For $A, B \in \mathsf{Ty}(\Gamma)$ with $E(0) = A$ and $E(1) = B$. Then there is a term*

$$\mathsf{equiv}(E) \in \mathsf{Weq}(A, B).$$

Proof. In $\mathsf{Ty}(\Gamma)$ we define the morphisms

$$f \colon A \to B, \quad f(x) := \mathsf{transp}(E, x),$$
$$g \colon B \to A, \quad g(y) := \mathsf{transp}(E, 1 - y),$$

and

$$u \colon \mathbb{I} \times A \to E, \quad u(i, x) := \mathsf{fill}(E(i), 0, x),$$
$$v \colon \mathbb{I} \times B \to E, \quad v(i, y) := \mathsf{fill}(E(1 - i), 0, y),$$

the latter of which can be seen as homotopies, satisfying

$$u(0, x) = x, \quad u(1, x) = f(x),$$

and

$$v(0, y) = g(y), \quad v(1, y) = y.$$

In fact, f is an equivalence. One can show that $(\Sigma x : A) P_B(y, f(x))$ is inhabited. One goes on showing that any two elements $\langle x_0, \beta_0\rangle$, $\langle x_1, \beta_1\rangle$ in this path space are connected by a path. This is done via composition and fillings (cf. [CCHM15, Sec. 7.1] for further details) using the homotopies defined above.
This suffices to show that f is an equivalence, hence, there is a term $\mathsf{equiv}(E) \in \mathsf{Weq}(A, B)$. $\square$

Theorem 3.3.2. *Given the universe* U *from Sec. 3.2, it has a composition structure.*

Proof. Let $E \in \mathsf{Ty}(\Gamma \times \mathbb{I})$ with $E(0) = A$ and $E(1) = B$.
Let $\overline{E}$ be the pullback along inversion:

$$\begin{array}{ccc} \overline{E} & \longrightarrow & E \\ \big\downarrow & \lrcorner & \big\downarrow \\ \Gamma \times \mathbb{I} & \xrightarrow[\mathrm{id}_\Gamma \times \neg]{} & \Gamma \times \mathbb{I} \end{array}$$

We define composition for the universe as

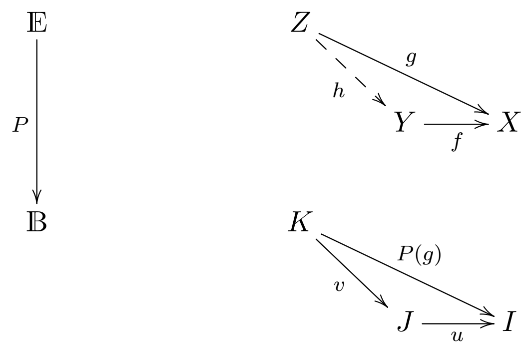
$$\mathsf{Comp}(\mathsf{U}, \varphi, E, A) := \mathsf{Glue}_\Gamma(\varphi, E(1), A, \mathsf{equiv}(\overline{E})) = \mathsf{Glue}_\Gamma(\varphi, B, A, \mathsf{equiv}(\overline{E}))$$

by means of Theorem 3.3.1. ∎

3.4 Interpreting Type Theory in Comprehension Categories

For categories $\mathbb{E}$ and $\mathbb{B}$, suppose we are given a functor $P\colon \mathbb{E} \to \mathbb{B}$.

Definition 3.4.1 (Cartesian Morphisms)**.** A morphism $f\colon Y \to X$ in $\mathbb{E}$ with $P(f) = u\colon J \to I$ is *(P-)cartesian* if for every $v\colon K \to J$ and $g\colon Z \to X$ with $P(g) = u \circ v$ there exists a unique $h\colon Z \to Y$ with $f \circ h = g$ and $P(h) = v$:



Definition 3.4.2 (Grothendieck Fibration)**.** A functor $P\colon \mathbb{E} \to \mathbb{B}$ is a *(Grothendieck) fibration* or a *fibered category* if for every $u\colon J \to I$ in $\mathbb{B}$ and $X \in \mathrm{ob}(\mathbb{E})$ with $P(X) = I$, there exists an object $X \in \mathrm{ob}(\mathbb{E})$ and a cartesian morphism $f\colon Y \to X$ in $\mathbb{E}$ such that $P(f) = u$. Such an f is called a *cartesian lifting of X along u.*

Definition 3.4.3 (Fibers and Vertical Morphisms)**.** Let $P\colon \mathbb{E} \to \mathbb{B}$ be a fibration. A morphism φ in $\mathbb{E}$ is called *vertical* if $P(\varphi) = \mathrm{id}_I$ for some $I \in \mathrm{ob}(\mathbb{B})$.
The *fiber of I* is the subcategory $P^{-1}(I) := \mathbb{E}_I \subset \mathbb{E}$ having as objects those $X \in \mathrm{ob}(\mathbb{E})$ such that $P(X) = I$ and as morphisms the vertical morphisms φ in $\mathbb{E}$ such that $P(\varphi) = \mathrm{id}_I$.

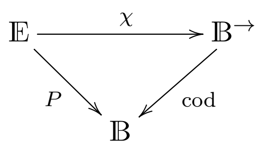
Definition 3.4.4 (Cloven and Split Fibrations)**.** A Grothendieck fibration $P\colon \mathbb{E} \to \mathbb{B}$ together with a choice of cartesian morphisms is called a *cloven fibration*: for every $u\colon J \to I$ and $X \in \mathrm{ob}(\mathbb{E})$ with $P(X) = I$, we have an object $u^*X \in \mathbb{E}$ and a cartesian morphism $\mathrm{Cart}(X)$.
A *split fibration* is a cloven fibration such that the chosen cleavage respects identities and composition:

(i) For all $I \in \mathrm{ob}(\mathbb{B})$ we have $\mathrm{id}_I^*(X) = X$ and $\mathrm{Cart}(\mathrm{id}_I) = \mathrm{id}_X$.

(ii) For all $u\colon J \to I$ and $v\colon K \to J$ and X with $P(X) = I$, we have $(u \circ v)^*X = u^*(v^*X)$ and $\mathrm{Cart}(u \circ v) = \mathrm{Cart}(u) \circ \mathrm{Cart}(v)$.

Example 3.4.5 (Codomain Fibration)**.** For a category $\mathbb{B}$, let $\mathbb{B}^{\to} := \mathbb{B} \downarrow \mathbb{B}$. If and only if $\mathbb{B}$ has pullbacks, the functor $\partial_1 := \mathrm{cod}\colon \mathbb{B}^{\to} \to \mathbb{B}$ mapping an object in $\mathbb{B}^{\to}$, i.e. an arrow in $\mathbb{B}$ to its codomain, is a fibration, called *fundamental fibration of* $\mathbb{B}$. The ∂_1-cartesian arrows are the pullbacks in $\mathbb{B}^{\to}$.

Definition 3.4.6 (Comprehension Category)**.** A *comprehension category* consists of categories $\mathbb{B}$ and $\mathbb{E}$ with a fibration $P\colon \mathbb{E} \to \mathbb{B}$ and a functor $\chi\colon \mathbb{E} \to \mathbb{B}^{\to}$ such that χ preserves the P-cartesian morphisms and P factors as follows:



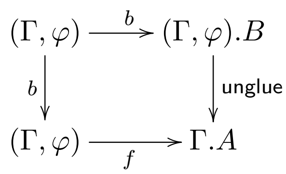
We call a comprehension category *full* if χ is full and faithful, and *split* if P is a split fibration.

In order to interpret type theory in a given comprehension category, the objects $\Gamma \in \mathrm{ob}(\mathbb{B})$ interpret the contexts. The fiber $P^{-1}(\Gamma)$ is taken to be $\mathsf{Ty}(\Gamma)$. For $A \in P^{-1}(\Gamma)$, the extended context $\Gamma.A$ is interpreted by $\mathrm{dom}(\chi(A))$. Terms of type A are modelled by sections of $\chi(A)$. The chosen cleavage of the fibration P hence corresponds to a choice of substitution functions for types.
In our case, let $\mathbb{B} = \mathbf{cSet}$ and $\mathbb{E}_\Gamma = \mathsf{Ty}(\Gamma)$ for $\Gamma \in \mathbb{B}$.

4 Univalence for Cubical Type Theory

4.1 The Univalence Axiom

Suppose we are given $f\colon T \to A$ and define $B := \mathsf{Glue}_\Gamma(\varphi, T, A, f)$. The map $\mathsf{unglue}\colon B \to A$ *extends* f in the following sense: For a term $b \in \mathsf{Ter}(\Gamma; A)$ we have:

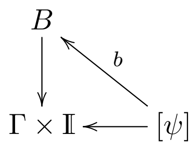


Theorem 4.1.1. *The map*

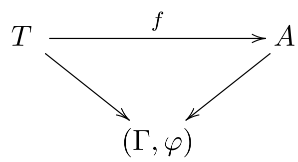
$$\mathsf{unglue}\colon \mathsf{Glue}_\Gamma(\varphi, T, A, f) \to A$$

is an equivalence.

Proof. Suppose we are given a term $u\colon \Gamma \to A$, a morphism $\psi\colon \Gamma \to \mathbb{F}$, a partial path b in B of extent $[\psi]$, i.e.



and a term $\alpha\colon (\Gamma, \psi) \to P_A(u, \mathsf{unglue}(b))$. Since



is a weak equivalence, using the equivalence operation, we get a term $t\colon [\varphi] \to T$ extending b, and a term β of path type $P_A(u, f(t))$.
Now, let $\tilde{a} := \mathsf{comp}_\Gamma(A, \varphi \vee \psi, \beta(i) \sqcup \alpha(i), u)$, $\tilde{b} := \mathsf{glue}(\varphi, t, \tilde{a})$ and $\tilde{\alpha} := \mathsf{fill}_\Gamma(A, \varphi \vee \psi, \beta(i) \sqcup \alpha(i), u)$. Then $\tilde{b}$ is a term of type B extending $\tilde{b}$ and $\tilde{\alpha}$ is a term of type $P_A(u, \mathsf{unglue}(\tilde{b}))$. □

Theorem 4.1.2. *Let $A, B \in \mathsf{Ty}(\Gamma)$. There is a map*

$$\mathsf{WeqToPath}_{A,B} =: \mathsf{WeqToPath}\colon \mathsf{Weq}(A, B) \to P_{\mathsf{U}}(A, B).$$

Proof. We define the map by

$$f \mapsto E_f = \mathsf{Glue}_\Gamma((i=0)\vee(i=1), A \sqcup B, B, f \sqcup \mathrm{id}_B).$$

□

Theorem 4.1.3. *For any $A : \mathsf{U}$, the type*

$$\mathsf{isContr}((\Sigma X : \mathsf{U})\mathsf{Weq}(X, A))$$

is inhabited.

Proof. It suffices to show that any partial element $\langle T, f\rangle$ of extent $[\varphi]$ of

$$C := (\Sigma X : U)\mathsf{Weq}(X, A)$$

can be connected to the restriction of a total element by a path.
Now, $\mathsf{unglue}(\varphi, T, f)$ extends f and is a weak equivalence. Since any two elements $\mathsf{isWeq}(X, A, \mathrm{pr}_1 \circ f)$ can be connected by a path, we conclude that any partial element of C can be connected by a path to the restriction of a total element. □

Theorem 4.1.4 (Univalence Axiom)**.** *For all $A, B \in \mathsf{Ty}(\Gamma)$, the canonical map*

$$\mathsf{PathToWeq} =: \mathsf{PathToWeq}_{A,B} \colon P_{\mathsf{U}}(A, B) \to \mathsf{Weq}(A, B)$$

is a weak equivalence.

Proof. By [Uni13, Thm. 4.7.7], we find that Thm. 4.1.3 is equivalent to the univalence axiom. □

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