THE SO(10)-GRAND UNIFIED THEORY

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Abstract. The SO(10) GUT is constructed as an extension of the SU(5) Theory and naturally acts on the whole of $\Lambda(\mathbb{C}^5)$ as a representation space. In particular, in this theory the laws of hypercharges from the Standard Model arise as simple consequences by assuming the existence of right-handed neutrinos.

We construct the necessary representations from the Spin groups in even dimension. Thus, we give a brief introduction into the structure and representation theory of Clifford algebras and Spin groups.

The account is mostly based on [BH10], [HBLM89, Ch. 1], [MFAS64] and [HN12, Ch. 17, Sec. 1; App. B, Sec. 3].

Contents

1. Clifford Algebras and Spin Groups

1.1. Clifford Algebras.

Let $k \in \{\mathbb{R}, \mathbb{C}\}$. In the following, all k-algebras are assumed finite-dimensional, associative and unital and with $\mathbf{1}$ denoting the respective one-element.

Definition 1.1 (Clifford Algebra)**.** Let V be a k-algebra and q a quadratic form on V. We consider the tensor algebra

$$T(V) := \bigoplus_{n=0}^{\infty} V^{\otimes r}$$

and the ideal $I_q \trianglelefteq T(V)$ generated by all elements of the form

$$v \otimes v + q(v) \cdot \mathbf{1}, \quad v \in V,$$

or equivalently

$$v \otimes w + w \otimes v = -2Q(v, w), \quad v, w \in V,$$

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with the polarization

$$Q(v,w) := \frac{1}{2}\left(q(v+w) - q(v) - q(w)\right).$$

The quotient algebra

$$\mathrm{Cl}(V,q) := T(V)/I_q$$

is called the *Clifford algebra of V with respect to q.*

There is a canonical embedding $\iota\colon V \rightarrowtail \mathrm{Cl}(V,q)$ given as the composition

$$\begin{tikzcd}
V = V^{\otimes 1} \arrow[r] \arrow[rr, bend left, "\iota"] & T(V) \arrow[r] & \mathrm{Cl}(V,q).
\end{tikzcd}$$

With this, the generating relations induce the following universal mapping property.

Theorem 1.2 (Universal Mapping Property)**.** *Let $f\colon V \to A$ be a k-linear map into a k-algebra A such that*

$$f(v)^2 = -q(v)\cdot \mathbf{1},\ v \in V.$$

Then there is a unique k-algebra homomorphism $\tilde{f}\colon \mathrm{Cl}(V,q) \to A$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Cl}(V,q) \arrow[r, dashed, "\tilde{f}"] & A \\
V \arrow[u, tail, "\iota"] \arrow[ur, "f"'] &
\end{tikzcd}$$

We give some lower-dimensional examples and also introduce the kind of real Clifford algebras we are going to work with in the following sections.

Example 1.3.

(1) If q is trivial, we can identify the Clifford algebra with the exterior algebra. We have k-linear isomorphism given by

$$\varphi\colon \mathrm{Cl}(V,0) \to \Lambda(V),\ \varphi(e_1 \cdot \ldots \cdot e_n) := e_1 \wedge \ldots \wedge e_n$$

for an orthonormal basis $\{e_1, \ldots, e_n\}$.

(2) Let V be one-dimensional and $e \in V$ a basis element. As an algebra, $\mathrm{Cl}(V,q)$ is generated by $\mathbf{1}$ and $x := \iota(e)$ where by definition $x^2 = q(e,e) =: a$, so $\mathrm{Cl}(V,q) \cong k[X]/(X^2 - a) =: \mathrm{Cl}(k,a)$. We now distinguish cases with respect to q:
   - If $q = 0$, then we obtain the ring of *dual numbers*, i.e. $\mathrm{Cl}(k,0) \cong k[X]/(X^2)$ which can be pictured as “first-order Taylor expansions” of formal polynomials.
   - If $q \neq 0$ and $a = b^2$ for some $b \in k$, then $x := b^{-1}e$ is a basis element of V such that $q(x,x) = 1$, so $\mathrm{Cl}(k,a) = k[X]/(X^2 - 1)$. For $c := \frac{1}{2}(\mathbf{1} + x)$ and $\overline{c} := \frac{1}{2}(\mathbf{1} - x)$ we obtain two idempotent elements in $\mathrm{Cl}(k,a)$ such that $c\overline{c} = 0$ and $c + \overline{c} = \mathbf{1}$, hence $\mathrm{Cl}(k,a) \cong k \oplus k$.
   - If on the other hand a is not a square in k, the polynomial $X^2 - a$ is irreducible over k and we obtain the splitting field $\mathrm{Cl}(k,a) \cong k[X]/(X^2 - a)$.

(3) If $k = \mathbb{R}$, we consider on $\mathbb{R}^{p+q}$ the graded standard scalar product given by

$$\langle v, w\rangle_{p,q} := \sum_{i=1}^{p} v_i w_i - \sum_{i=p+1}^{p+q} v_i w_i.$$

The corresponding Clifford algebra is denoted by

$$\mathrm{Cl}_{p,q} := \mathrm{Cl}\big(\mathbb{R}^{p+q}, \langle v, w\rangle_{p,q}\big).$$

However, in our cases of application, it suffices to stick to the definite algebras

$$\mathrm{Cl}_n := \mathrm{Cl}_{n,0}.$$

(4) Using the above considerations we find $\mathrm{Cl}_1 \cong \mathbb{C}$, $\mathrm{Cl}_{0,1} \cong \mathbb{R} \oplus \mathbb{R}$. Furthermore $\mathrm{Cl}_{2,0} \cong \mathbb{H}$, where $\mathbb{H}$ denotes the skew-field of quaternions, and $\mathrm{Cl}_{0,2} \cong \mathrm{Cl}_{1,1} \cong M_2(\mathbb{R})$ (cf. [HBLM89, Ch. 1, Sec. 4]).

The dimension of a Clifford algebra is given by $\dim \mathrm{Cl}(V,q) = 2^{\dim(V)}$.

The next consideration is important for the study of the structure and hence the representations of Clifford algebras.

Definition 1.4 (Involution and grading of $\mathrm{Cl}(V,q)$)**.** By the universal mapping property 1.2 there is a unique involutive automorphism $\omega\colon \mathrm{Cl}(V,q) \to \mathrm{Cl}(V,q)$ such that

$$\omega \circ \iota = -\iota,$$

called the *grading automorphism.* This is due to the reason that the eigenspaces

$$\mathrm{Cl}(V,q)_0 := \ker(\omega - \mathbf{1}) \quad \text{and} \quad \mathrm{Cl}(V,q)_1 := \ker(\omega + \mathbf{1})$$

introduce a $\mathbb{Z}/2\mathbb{Z}$-grading on $\mathrm{Cl}(V,q)$, i.e.

$$\mathrm{Cl}(V,q) \cong \mathrm{Cl}(V,q)_0 \oplus \mathrm{Cl}(V,q)_1$$

and

$$\mathrm{Cl}(V,q)_a \cdot \mathrm{Cl}(V,q)_b \subset \mathrm{Cl}(V,q)_{a+b},\ a,b \in \mathbb{Z}/2\mathbb{Z}.$$

1.2. Clifford Groups.

Definition 1.5 (Twisted adjoint representation and Clifford group)**.** The *twisted adjoint representation* of the unit group $\mathrm{Cl}(V,q)^\times$ on the algebra $\mathrm{Cl}(V,q)$ is defined by

$$\mathrm{Ad}\colon \mathrm{Cl}(V,q)^\times \times \mathrm{Cl}(V,q) \to \mathrm{Cl}(V,q),\ (a,x) \mapsto \mathrm{Ad}(a)x := \omega(a)xa^{-1}.$$

The stabilizer of the subspace $V \cong \iota(V) \subset \mathrm{Cl}(V,q)$ is called the *Clifford group*

$$\Gamma(V,q) := \left\{a \in \mathrm{Cl}(V,q)^\times \colon \mathrm{Ad}(a)V = V\right\}.$$

The twisted adjoint representation of the unit group induces a representation of the Clifford group

$$\Phi\colon \Gamma(V,q) \to \mathrm{GL}(V), a \mapsto \mathrm{Ad}(a)|_V.$$

Again, the universal mapping property of Clifford algebras 1.2 implies the unique existence of an *anti-involution* $(-)^*\colon \mathrm{Cl}(V,q) \to \mathrm{Cl}(V,q)$, i.e. an involution which is an *antiautomorphism*, meaning $(xy)^* = y^*x^*$ for all $x,y \in \mathrm{Cl}(V,q)$ satisfying $v^* = -v$ for $v \in V$. Furthermore $\omega \circ (-)^* = (-)^* \circ \omega$ (cf. [HN12, Lemma B.3.11].

Example 1.6.

(1) On $\mathrm{Cl}_1 \cong \mathbb{C}$, the anti-involution is just conjugation, so $\mathrm{Ad}(z)w = \overline{z}wz^{-1} = \overline{z}z^{-1} \cdot w$ and

$$\Gamma(\mathrm{Cl}_1) = \left\{z \in \mathbb{C}^\times \colon \overline{z}z^{-1} = \frac{\overline{z}^2}{|z|^2} \in \mathbb{R}\right\} = \mathbb{R}^\times 1 \sqcup \mathbb{R}^\times \mathrm{i}.$$

(2) There is a similar result for the quaternions $\mathrm{Cl}_2 \cong \mathbb{H} = \langle 1, I, J, K\rangle$. In this case

$$\Gamma(\mathrm{Cl}_2) = \mathbb{R}^\times\{\alpha \cdot 1 + \delta K \colon \alpha, \delta \in \mathbb{R},\ \alpha^2 + \delta^2 = 1\} \sqcup \mathbb{R}^\times\{\beta I + \gamma J \colon \beta, \gamma \in \mathbb{R},\ \beta^2 + \gamma^2 = 1\}.$$

Lemma 1.7. *The Clifford group* $\Gamma(V,q)$ *is invariant under* ω *and* $(-)^*$.

Proof. Let $g \in \Gamma(V, q)$ and $v \in V$. Then $\mathrm{Ad}(g)v = \omega(g)vg^{-1} \in V$ leads to

$$V \ni \mathrm{Ad}(g)v = -\omega\left(\mathrm{Ad}(g)v\right) = -g\omega(v)\omega(g)^{-1} = \mathrm{Ad}\left(\omega(g)\right)v,$$

so $\omega(g) \in \Gamma(V, q)$. Analgously

$$V \in \mathrm{Ad}(g)v = -\left(\mathrm{Ad}(g)v\right)^* = -(g^*)^{-1}v^*\omega(g^*) = \mathrm{Ad}(\omega(g^*)^{-1})v \in V,$$

so $\omega(g^*) \in \Gamma(V, q)$ and hence $g^* \in \Gamma(V, q)$. □

Theorem 1.8. *Let V be a finite-dimensional vector space and q a non-degenerate form on V. Then the kernel of the representation $\Phi\colon \Gamma(V, q) \to \mathrm{GL}(V)$ is $k^\times \mathbf{1}$.*

Proof. See [HN12, Lemma B.3.13]. □

Definition 1.9 (Clifford norm)**.** We define the *Clifford norm* of the algebra $\mathrm{Cl}(V, q)$ by

$$N\colon \mathrm{Cl}(V, q) \to \mathrm{Cl}(V, q),\ x \mapsto xx^*.$$

Theorem 1.10. *If $x \in \Gamma(V, q)$, we have $N(x)\mathbf{1} \in K^\times \mathbf{1}$, so the Clifford norm $N\colon \Gamma(V, q) \to \mathbb{R}^\times$ is a homomorphism. Moreover,*

$$N(\omega(g)) = N(g) \quad \textit{and} \quad N(\mathrm{Ad}(g)h) = N(h) \quad \textit{for} \quad g, h \in \Gamma(V, q).$$

Proof. As $\Gamma(V, q)$ is invariant under $(-)^*$, we have $gg^* \in \Gamma(V, q)$. By the foregoing theorem, $gg^* \in \ker \varphi$ will imply $gg^* \in k^\times \mathbf{1}$, so we are going to show that the precondition is indeed satisfied.

We define an involutive antiautomorphism on $\mathrm{Cl}(V, q)$ by $S(x) := \omega(x^*)$. Then S fixes V pointwise and, since $\Gamma(V, q)$ is invariant under $(-)^*$, we have to show that $\Phi(g^{-1}) = \Phi(g^*)$ for $g \in \Gamma(V, q)$. If $g \in \Gamma(V, q)$ and $v \in V$, the element $\Phi(g^*)v = \omega(g^*)v(g^{-1})^* = S(g)v(g^{-1})^* \in V$ is fixed by S, so

$$\Phi(g^*)v = S(S(g)v(g^{-1})^*) = S((g^{-1})^*)vg,$$

so $\Phi(g^*) = \Phi(g^{-1})$ and consequently $gg^* \in \ker \Phi = k^\times \mathbf{1}$. This means, we can define N the desired way.

We yet have to check that N is a homomorphism. We calculate

$$N(gh)\mathbf{1} = ghh^*g^* = g(N(h)\mathbf{1})g^* = N(h)gg^* = N(h)N(g)\mathbf{1}.$$

The remaining relation are verified as follows. Applying ω to $gg^* = N(g)\mathbf{1}$ yields $N(\omega(g))\mathbf{1} = \omega(g)\omega(g)^* = N(g)\mathbf{1}$, so $N(\omega(g)) = N(g)$ and thus $N(\mathrm{Ad}(g)h) = N(\omega(g)hg^{-1}) = N(h)$. □

Theorem 1.11. *If V is a finite-dimensional vector space and q a non-degenerate form on V, then the image of the representation Φ is given by the* orthogonal group

$$\mathrm{im}(\Phi) = \mathrm{O}(V, q) := \{\alpha\colon \mathrm{GL}(V)\colon \alpha^* \circ q = q\}.$$

This yields a short exact sequence

$$1 \to K^\times \rightarrowtail \Gamma(V, q) \xrightarrow{\Phi} \mathrm{O}(V, \beta) \to 1$$

where Φ acts as reflection

$$\Phi(v)x = x - 2\frac{q(v, x)}{q(v, v)}v, x \in \Gamma(V, q), x \in \mathrm{Cl}(V, q)$$

for non-isotropic vectors $v \in V \subset \mathrm{Cl}(V, q)$, *i.e.* $q(v, v) \neq 0$.

Proof. For $v \in V$, by the generating relation for the Clifford algebra, we have

$$vv^* = -v^2 = -q(v,v)\mathbf{1},$$

and for $g \in \Gamma(V,q)$ we have

$$\begin{aligned}(\Phi(g)v)(\Phi(g)v)^* &= \omega(g)vg^{-1}(\omega(g)vg^{-1})^* = \omega(g)vg^{-1}(g^{-1})^*(-v)\omega(g)^* \\ &= -N(g^{-1})q(v,v)\omega(gg^*) = -N(g^{-1})q(v,v)\omega(N(g)\mathbf{1}) = -q(v,v)\mathbf{1}.\end{aligned}$$

Again, by the defining relation, we have $\Phi(g) \in \mathrm{O}(V,q)$.

We now investigate the image of Φ. Let $v \in V$ be non-isotropic, so $\omega(v) = -v$ and $v^{-1} = q(v,v)^{-1}v$. This implies

$$\mathrm{Ad}(v)x = -q(v,v)^{-1}vxv = q(v,v)^{-1}v(vx - 2q(v,x)\mathbf{1}) = x - 2\frac{q(v,x)}{q(v,v)}v =: \sigma_v,$$

i.e. the adjoint representation acts as the orthogonal reflection in the hyperplane $\{v\}^\perp$. In particular, $\Gamma(V,q)$ contains all non-isotropic vectors of $V \subset \mathrm{Cl}(V,q)$ and $\mathrm{im}\Phi$ contains all orthogonal reflections. In the case of V being finite-dimensional and q being non-degenerate, all of $\mathrm{O}(V,q)$ is indeed generated by reflections, so indeed $\mathrm{im}(\Phi) = \mathrm{O}(V,q)$. □

In particular, we have thus shown that $\Gamma(V,q)$ is generated by the set $\ker(\Phi) \cup \{v \in V : q(v,v) \neq 0\}$, since $\mathrm{im}(\Phi)$ is generated by the orthogonal reflections $\Phi(v)$.

Example 1.12. In the case of $\mathrm{Cl}(V,q) \cong \Lambda(V)$ (i.e. $q = 0$), we have $\Lambda(V)^\times = k^\times\mathbf{1} \oplus \bigoplus_{k=1}^\infty \Lambda^k(V)$. As $\Lambda(V)$ is graded commutative, the even part is central and any two odd elements anticommute. Decomposing any $g \in \Lambda(V)^\times$ $g = g_+ + g_-$ where g_+ is even and g_- is odd, we have $g_+v = vg_+$ and $g_-v = -vg_-$ for all $v \in V$. This means $\omega(g)v = vg$, so $\Lambda(V)^\times = \Gamma(V,q) = \ker\Phi$.

Theorem 1.13. *The topological subgroup* $\Gamma(V,\beta) \leq \mathrm{O}(V,q)$ *is a Lie group.*

Proof. This follows, since $\mathrm{O}(V,q)$ is closed. □

1.3. **Pin and Spin Groups.**

Definition 1.14 (Pin and Spin groups)**.** The kernel of the norm homomorphism $N\colon \Gamma(V,q) \to k^\times$ is called the *Pin group*

$$\mathrm{Pin}(V,q) := \ker(N)$$

. The subgroup consisting of the even elements is called the *Spin group*

$$\mathrm{Spin}(V,q) := \mathrm{Pin}(V,q) \cap \mathrm{Cl}_0(V,q).$$

Example 1.15. In the case of the Clifford algebra $\mathrm{Cl}_1 \cong \mathbb{C}$, $\omega(z) = \bar{z}$, the Clifford group is $\mathbb{R}^\times \cup \mathrm{i}\mathbb{R}^\times$. By $N(z) = |z|^2$, we have $\mathrm{Pin}_1(\mathbb{R})\{\pm 1, \pm\mathrm{i}\}$ and $\mathrm{Spin}_1(\mathbb{R}) = \{\pm 1\}$. For the quaternions one can show that $\mathrm{Pin}_2(\mathbb{R}) \cong \mathrm{S}^1$.

Theorem 1.16. *If* $k = \mathbb{R}$ *and* q *is positive definite, there are short exact sequences*

$$1 \to \mathbb{Z}/2\mathbb{Z} \to \mathrm{Pin}(V,q) \to \mathrm{O}(V,q) \to 1$$

and

$$1 \to \mathbb{Z}/2\mathbb{Z} \to \mathrm{Spin}(V,q) \to \mathrm{SO}(V,q) \to 1.$$

Proof. For each non-zero $v \in V$ we have $N(v) = q(v,v) > 0$, so for $v' := \frac{v}{N(v)}$, we have $v' \in \mathrm{Pin}(V,q)$. Hence, the restriction $\Phi\colon \mathrm{Pin}(V,q) \to \mathrm{O}(V,q)$ is still surjective. This yields the desired exact sequences. □

Example 1.17. In fact, for general q, the homomorphism $\Phi\colon \mathrm{Spin}(V,q) \to \mathrm{SO}(V,q)$ need not be surjective. We choose $v_1, v_2 \in V$ such that $q(v_1,v_1) = 1 = -q(v_2,v_2)$. The composition of the reflections $g := \sigma_{v_1}\sigma_{v_2}$ is in $\mathrm{SO}(V,q)$. In $\Gamma(V,q)$ we have $N(v_1v_2) = N(v_1)N(v_2) = -1$. Then $\Phi(v_1v_2) = g$ and for any element $\gamma \in \Phi^{-1}(g)$, we have $\gamma = \lambda v_1 v_2$, $\lambda \in k^\times$, so $N(\gamma) = -\lambda^2 < 0$ and consequently $\gamma \notin \mathrm{Spin}(V,q)$. So in this case, $\Phi(\mathrm{Spin}(V,q))$ is a proper subgroup of $\mathrm{SO}(V,q)$.

Theorem 1.18. *The restriction of* Φ *to* $\mathrm{Spin}_n(\mathbb{R})$ *is a double covering with discrete kernel* $\{\pm 1\}$. *We have a short exact sequence:*

$$1 \to \mathbb{Z}/2\mathbb{Z} \to \mathrm{Spin}_n(\mathbb{R}) \to \mathrm{SO}(n) \to 1$$

Also, $\mathrm{Spin}_n(\mathbb{R})$ *is connected.*

Proof. One can show that $\mathrm{SO}(n)$ is connected and its fundamental group has at most two elements, cf. [HN12, Prop. 17.1.9]. We only have to show that $\mathrm{Spin}_n(\mathbb{R})$ is connected, i.e. $-\mathbf{1} \in \mathrm{Pin}_n(\mathbb{R})_0$. We identify the basis elements $\{e_1, \ldots, e_n\} \subset \mathbb{R}^n$ with the corresponding elements of Cl_n and set

$$\gamma(t) := \cos(t)\mathbf{1} + \sin(t)e_1e_2.$$

Now $(e_1e_2)^2 = -\mathbf{1}$, so $\gamma(t) = e^{te_1e_2}$ which using the grading automorphism ω implies $\omega(\gamma(t)) = \gamma(t)$ and $\gamma(t)^{-1} = \gamma(-t)$. This means

$$\omega(\gamma(t))e_1\gamma(t)^{-1} = \cos(2t)e_1 + \sin(2t)e_2$$
$$\omega(\gamma(t))e_1\gamma(t)^{-1} = -2\sin(2t)e_1 + \cos(2t)e_2$$
$$\omega(\gamma(t))e_i\gamma(t)^{-1} = e_i, i \geq 3.$$

Hence γ is an element of the Clifford group. Now $\gamma(t)\gamma(t)^* = \gamma(t)\gamma(-t) = \mathbf{1}$ and so $\gamma(t) \in \mathrm{Pin}_n(\mathbb{R})$. Finally, $\gamma(\pi) = -\mathbf{1}$ yields the claim. □

In particular the special case

$$1 \to \mathbb{Z}/2\mathbb{Z} \to \mathrm{Spin}_n \to \mathrm{SO}(n) \to 1$$

shows that $\mathrm{Spin}_n \to \mathrm{SO}(n)$ is the universal covering of $\mathrm{SO}(n)$ for $n \geq 3$ with $\pi_1(\mathrm{SO}(n)) = \mathbb{Z}/2\mathbb{Z}$.

2. The Dirac Spinor Representation

2.1. The Dirac Spinor Representation.

On $\mathbb{C}^n$, we consider the default Hermitian scalar product

$$\langle v, w\rangle := \sum_{i=1}^n v_i\overline{w_i}.$$

Definition 2.1 (Contraction)**.** For $a \in \mathbb{C}^n$, we define the so-called *contraction by* a on $\Lambda^k(\mathbb{C}^n)$ for $k \geq 1$ by

$$\iota_a\colon \Lambda^k(\mathbb{C}^n) \to \Lambda^{k-1}(\mathbb{C}^n)$$
$$\iota_a(v_1 \wedge \ldots \wedge v_k) := \sum_{i=1}^k,(-1)^{i+1}\langle v_i, a\rangle v_1 \wedge \ldots \wedge \widehat{v_i} \wedge \ldots \wedge v_k.$$

Since $\iota_a \circ \iota_a = 0$, the universal property of the exterior product implies that we can lift these contractions to a unique algebra homormophism $\iota_\omega\colon \Lambda(\mathbb{C}^n) \to \Lambda(\mathbb{C}^n)$ for arbitrary $\omega \in \Lambda(\mathbb{C}^n)$.

In the case of $\{e_1, \ldots, e_n\}$ being an orthonormal basis for $\mathbb{C}^n$, for $1 \leq i \leq n$, we write $\iota_{a_i} =: a_i\colon \Lambda(\mathbb{C}^n) \to \Lambda(\mathbb{C}^n)$ and call a_i the *annihilation operator for* e_i.

Definition 2.2 (Multiplication)**.** For $\omega \in \Lambda(\mathbb{C}^n)$, we define the so-called *multiplication by* ω as

$$\mu_\omega \colon \Lambda^k(\mathbb{C}^n) \to \Lambda^{k-1}(\mathbb{C}^n), \quad \mu_\omega(\eta) := \omega \wedge \eta.$$

In the case of $\{e_1, \dots, e_n\}$ being an orthonormal basis for $\mathbb{C}^n$, for $1 \leq i \leq n$, we write $a_i^* =: \mu_{e_i} \colon \Lambda(\mathbb{C}^n) \to \Lambda(\mathbb{C}^n)$ and call a_i^* the *creation operator for* e_i.

In fact, the operators ι_ω and μ_ω are adjoint w.r.t. to the scalar product defined on the exterior algebra by a dual pairing and identification of dual spaces (cf. [War89, pp. 59] for further details).

The terms "annihilation" and "creation operators" come from the physical modelling of particles as vectors e_j. The operator a_j^* creates a particle of type j in a configuration and a_j deletes it.

We denote the complexified exterior product by $\Lambda_{\mathbb{C}}(\mathbb{C}^n) := \Lambda(\mathbb{C}^n) \otimes_{\mathbb{R}} \mathbb{C}$. For $v \in \mathbb{C}^n$, we define the map

$$f_v \colon \Lambda_{\mathbb{C}}(\mathbb{C}^n) \to \Lambda_{\mathbb{C}}(\mathbb{C}^n),\ f_v(\omega) := (\mu_v - \iota_v)(\omega) = v \wedge \omega - \iota_v(\omega).$$

One can verify that contraction is an anti-derivation, i.e.

$$\iota_\alpha(\omega \wedge \eta) = \iota_\alpha(\omega) \wedge \eta + (-1)^k \omega \wedge \iota_\alpha(\eta) \tag{1}$$

if $\omega \in \Lambda^k(\mathbb{C}^n)$. Using this, we compute

$$\begin{aligned}(f_v \circ f_v)(\omega) &= \underbrace{v \wedge v \wedge \omega}_{=0} - v \wedge \iota_v(\omega) - (\iota_v(v \wedge \omega) - \underbrace{\iota_v(\iota_v(\omega)))}_{=0} \\ &= -v \wedge \iota_v(\omega) - (\langle v, v\rangle \omega + v \wedge \iota_v(\omega)) \\ &= -\langle v, v\rangle \omega.\end{aligned}$$

The map

$$f \colon \mathbb{C}^n \to \mathrm{End}_{\mathbb{R}}(\Lambda_{\mathbb{C}}(\mathbb{C}^n)) \cong \mathrm{End}_{\mathbb{C}}(\Lambda(\mathbb{C}^n)), \quad v \mapsto f_v$$

is $\mathbb{R}$-linear. Now, by $\mathbb{C}^n \cong \mathbb{R}^{2n}$ and the universal property of Clifford algeras, we see that the property $f_v \circ f_v = -\langle v, v\rangle$ defines a unique extension of f to a representation

$$\pi \colon \mathrm{Cl}_{2n} \to \mathrm{End}_{\mathbb{C}}(\Lambda(\mathbb{C}^n)).$$

Since π has complex dimension $\dim_{\mathbb{C}} \Lambda(\mathbb{C}^n) = 2^n$, so by [HBLM89, Thm. 5.7], it is the unique irreducible representation of Cl_{2n}.

Definition 2.3 (Dirac Spinor Representation)**.** Since $\mathrm{Spin}_{2n} \subset \mathrm{Cl}_{2n}$ we can restrict the representation defined above to Spin_{2n}, yielding the *Dirac spinor representation*

$$\rho' := \pi|_{\mathrm{Spin}_{2n}} \colon \mathrm{Spin}_{2n} \to \mathrm{End}_{\mathbb{C}}(\Lambda(\mathbb{C}^n)).$$

2.2. **Extending the Representation of** SU(n)**.**

In fact the image of the Dirac spinor representation restricts to the unitary endomorphisms $\mathrm{U}(\Lambda(\mathbb{C}^n) \subset \mathrm{End}_{\mathbb{C}}(\Lambda(\mathbb{C}^n))$. Hence, we can prove the following central result which shows that the Dirac spinor representation extends the standard representation of $\mathrm{SU}(n)$ on $\Lambda(\mathbb{C}^n)$.

Theorem 2.4. *There exists a morphism* $\psi \colon \rho \to \rho'$ *of Lie group representations, i.e.* $\psi \colon \mathrm{SU}(n) \to \mathrm{Spin}_{2n}$ *is a Lie group morphism making the diagram*

$$\begin{tikzcd} \mathrm{SU}(n) \arrow[r, "\psi"] \arrow[d, "\rho"'] & \mathrm{Spin}_{2n} \arrow[dl, "\rho'"] \\ \mathrm{U}(\Lambda(\mathbb{C}^n)) & \end{tikzcd} \tag{2}$$

commute.

Proof. The proof is due to [BH10, Thm. 2].

Using the isomorphism $\mathbb{C}^n \cong \mathbb{R}^{2n}$ and the real part of the default hermitian scalar product on $\mathbb{C}^n$ yields an inclusion $\mathrm{U}(n) \rightarrowtail \mathrm{O}(2n)$. The connected component of $E \in \mathrm{O}(2n)$ is $\det^{-1}(\{1\}) = \mathrm{SO}(2n)$. Since $\mathrm{U}(n)$ is connected as well, we have an inclusion $\mathrm{U}(n) \rightarrowtail \mathrm{SO}(2n)$, and also $\mathrm{SU}(n) \rightarrowtail \mathrm{SO}(2n)$. This induces an injective Lie algebra morphism $\mathfrak{su}(n) \rightarrowtail \mathfrak{so}(2n)$. This Lie algebra morphism can be uniquely integrated to a Lie group morphism between the corresponding simply-connected Lie groups, so we get a Lie group morphism $\psi\colon \mathrm{SU}(n) \to \mathrm{Spin}_{2n}$ (for example cf. [HN12, Cor. 9.5.10]).

We have to show that ψ is really a morphism of representations. Since all the groups $\mathrm{SU}(n), \mathrm{Spin}_{2n}, \mathrm{U}(\Lambda(\mathbb{C}^n))$ are connected, by integration, it suffices to show check the analogous claim or adjoint the morphism $\mathrm{d}\psi$ on the level of Lie algebras, i.e. the following diagram commutes:

$$\begin{tikzcd}
\mathfrak{su}(n) \arrow[r, "\mathrm{d}\psi"] \arrow[d, "\mathrm{d}\rho"'] & \mathfrak{so}(2n) \arrow[dl, "\mathrm{d}\rho'"] \\
\mathfrak{u}(\Lambda(\mathbb{C}^n)) &
\end{tikzcd} \tag{3}$$

Since each element from $\mathfrak{su}(n)$ has vanishing trace, a basis of $\mathfrak{su}(n)$ is given by the elements:

$$\begin{array}{lr}
E_{jk} - E_{kj} & k < j \\
\mathrm{i}(E_{jk} + E_{kj}) & k < j \\
\mathrm{i}(E_{jj} - E_{j+1,j+1}) & j = 1, \ldots, n-1
\end{array}$$

where E_{jk} has 1 at position (j,k) and 0 elsewhere.

Since $E_{jk} \cdot e_l = \delta_{lk} e_j$ for basis elements e_l of $\mathbb{C}^n \cong \Lambda^1(\mathbb{C}^n)$, the matrices E_{jk} act on $\Lambda^1(\mathbb{C}^n)$ the same way as the composed operators $a_j^* a_k$. Hence, on $\Lambda^1(\mathbb{C}^n)$ we get the formulas

$$\mathrm{d}\rho(E_{jk} - E_{kj}) = a_j^* a_k - a_k^* a_j, \tag{4}$$
$$\mathrm{d}\rho(\mathrm{i}(E_{jk} + E_{kj})) = \mathrm{i}(a_j^* a_k - a_k^* a_j), \tag{5}$$
$$\mathrm{d}\rho(\mathrm{i}(E_{jj} - E_{j+1,j+1})) = \mathrm{i}(a_j^* a_j - a_{j+1}^* a_{j+1}). \tag{6}$$

In the following, we want to show that $\mathrm{d}\rho$ is defined accordingly on the whole algebra $\Lambda(\mathbb{C}^n)$.

Since for $\rho\colon \mathrm{SU}(n) \to \mathrm{U}(\Lambda(\mathbb{C}^n))$ and $x \in \mathrm{SU}(n)$ is an algebra morphism by definition, we have

$$\rho(x)(\omega \wedge \eta) = \rho(x)\omega \wedge \rho(x)\eta,$$

so

$$\mathrm{d}\rho(X)(\omega \wedge \eta) = (\mathrm{d}\rho(X)\omega) \wedge \eta + \omega \wedge (\mathrm{d}\rho(X)\eta)$$

for all $X \in \mathfrak{su}(n)$. This means $\mathrm{d}\rho$ is really a derivation-valued representation. Now, derivations of $\Lambda(\mathbb{C}^n)$ are uniquely determined by their values on $\Lambda^1(\mathbb{C}^n)$, so we are to show that the values of $\mathrm{d}\rho$ on the basis given above are really derivations. This is in fact true, since the composites $a_j^* a_k$ are derivations. To see this, we at first recall that the annihilation operators a_k are antiderivations (cf. 2.1) and that for the creation operators it holds that $a_j^*(\omega \wedge \eta) = a_j^* \omega \wedge w =$

$(-1)^p \omega \wedge a_j^* \eta$ for $\omega \in \Lambda^p(\mathbb{C}^n)$. We compute

$$\begin{aligned} a_j^* a_k(\omega \wedge \eta) &= a_j^*((a_k\omega) \wedge \eta + (-1)^p \omega \wedge (a_k \eta)) \\ &= (a_j^* a_k)(\omega) \wedge \eta + (-1)^p (a_j^* \omega) \wedge \eta \\ &= (a_j^* a_k)(\omega) \wedge \eta + \underbrace{(-1)^{2p}}_{=1} \omega \wedge (a_j^* a_k)(\eta), \end{aligned}$$

so indeed $a_j^* a_k$ are derivations.

Now, since by definition ψ is inclusion and $\rho(v)(\omega) = v \wedge \omega - \iota_v(\omega)$, invoking the formulas 4, the factorization $\mathrm{d}\rho = \mathrm{d}\rho' \circ \mathrm{d}\psi$ holds. ☐

3. The Spin(10) GUT

From the preceding chapter, we know the SU(5) GUT is given by the diagram:

$$\begin{tikzcd} G_{\mathrm{SM}} \arrow[r, "\varphi"] \arrow[d] & \mathrm{SU}(5) \arrow[d] \\ \mathrm{U}(F \oplus F^*) \arrow[r, "\mathrm{U}(f)"'] & \mathrm{U}(\Lambda(\mathbb{C}^5)) \end{tikzcd} \tag{7}$$

Horizontal composition with the diagram

$$\begin{tikzcd} \mathrm{SU}(5) \arrow[r, "\psi"] \arrow[d, "\rho"] & \mathrm{Spin}(10) \arrow[dl, "\rho'"] \\ \mathrm{U}(\Lambda(\mathbb{C}^5)) & \end{tikzcd} \tag{8}$$

yields:

$$\begin{tikzcd} G_{\mathrm{SM}} \arrow[r, "\psi\circ\varphi"] \arrow[d] & \mathrm{Spin}(10) \arrow[d] \\ \mathrm{U}(F \oplus F^*) \arrow[r, "\mathrm{U}(f)"'] & \mathrm{U}(\Lambda(\mathbb{C}^5)) \end{tikzcd} \tag{9}$$

So, Spin(10) extends the Standard Model representation of SU(5) yielding the Spin(10) GUT.

One can further analyze how the representation of Spin($2n$) on $\Lambda(\mathbb{C}^n)$ decomposes into two irreducible subrepresentations w.r.t. the grading of $\Lambda(\mathbb{C}^n)$. Elements of the sub-representation spaces are accordingly called left- or right-handed Weyl spinors. They play a role in the analysis of massless particles of spin 1/2 within Relativistic Quantum Field Theory.

References

[BH10] John C. Baez and John Huerta. The Algebra of Grand Unified Theories. *Bull.Am.Math.Soc.*, 47:483–552, 2010.

[HBLM89] Jr. H. Blaine Lawson and Marie-Louise Michelsohn. *Spin Geometry.* Princeton University Press, 1989.

[HN12] Joachim Hilgert and Karl-Hermann Neeb. *Structure and Geometry of Lie Groups.* Springer Monographs in Mathematics. Berlin: Springer, 2012.

[MFAS64] Raoul Bott Michael F. Atiyah and Arnold Shapiro. Clifford Modules. *Topology*, 3:pp. 3–38, 1964.

[War89] Frank W. Warner. *Foundations of Differentiable Manifolds and Lie Groups.* Springer, 1989.