Locally Compact Groups: Topological Aspects of the Isomorphism Theorems & Cyclic Subgroups

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Contents

Abstract

In Section 1 we prove several classical isomorphism theorems for topological groups. Furthermore, we state sufficient criteria for a topological group to be isomorphic to an inner direct product. In order to do so, we will need an open mapping theorem for topological groups which yields that every surjective morphism between topological groups is open if the groups satisfy certain compactness properties.

We proceed in Section 2 by analyzing the structure of certain locally compact groups based on their subgroups. Weil's Lemma consists of two structure results for locally compact Hausdorff groups G. In particular, for each $g \in G$ the cyclic group $\langle g \rangle$ is either discrete and infinite or has compact closure in G. We continue by classifying certain Abelian topological groups as direct products of a free Abelian group with an open subgroup. Additionally, we state an existence criterion for discrete subgroups of locally compact Abelian Hausdorff groups. Finally, we give some results of purely algebraic nature.

This treatise was prepared for the seminar "Locally Compact Groups" held by PD. Dr. Ralf Gramlich in August 2010 at TU Darmstadt. The seminar was structured according to Markus Stroppel's book [3]. Further resources are provided under `http://www3.mathematik.tu-darmstadt.de/index.php?id=84&evsid=23&&evsver=880`.

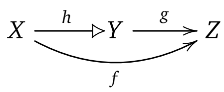
Notation

The mappings will be denoted as actions from the right. The image of a point x under a map f will be written x^f. Composition of mappings transforms likewisely, i.e. $x^{(f \circ g)} := \left(x^f\right)^g$.

1 Topological Aspects of the Isomorphism Theorems

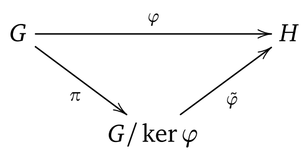
Definition 1.1 (Quotient Map)**.** Let $f: X \to Y$ be a surjective map between topological spaces. We call f a *quotient map*, if it induces the quotient topology on Y. This means every subset $U \subset Y$ is open if and only $U^{f^{-1}} \subset X$ is open.

Lemma 1.2 (Universal Mapping Property of Quotient Maps)**.** *We consider maps between topological spaces $h: X \to Y$ and $g: Y \to Z$ with $f := h \circ g$. If h is a quotient map and f is continuous, then g is also continuous. Hence, we have the following situation:*



Proof. Let $O \subset Z$ be open. As f is continuous, $O^{f^{-1}}$ is open. By definition of the mappings $O^{g^{-1}h^{-1}} = O^{f^{-1}}$. As h induces the quotient topology on Y and $O^{g^{-1}h^{-1}}$ is open, also $O^{g^{-1}}$ is open. □

Theorem 1.3 (Homomorphism Theorem for Topological Groups [2])**.** *Let $\varphi: G \to H$ be a morphism of topological groups and $\pi: G \to G/\ker\varphi$ the natural projection. Then there exists a uniquely determined bijective morphism of topological groups $\tilde{\varphi}: G/\ker\varphi \to \operatorname{im}\varphi$ such that the diagram*



is commutative.

Proof. The isomorphism theorem for groups yields the desired bijective group morphism $\tilde{\varphi}$. As shown by [3, Lemma 6.2(a)], the projection π is open. From Lemma 1.2 we derive that $\tilde{\varphi}$ is continuous. □

Example 1.4 (The Torus as a Quotient)**.** Let $(\mathbb{T}, \cdot)$ be the 1-torus. The kernel of the surjective morphism

$$\mathbb{R} \to \mathbb{T}, \ t \mapsto \exp(2\pi i t)$$

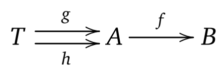
is $\mathbb{Z}$, so we obtain a *continuous* group isomorphism $\mathbb{R}/\mathbb{Z} \cong \mathbb{T}$.

This is actually an isomorphism of topological groups as we prove later on by means of an open mapping theorem for locally compact groups.

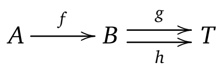
The following lemma is needed for a technical argument in the proof of the upcoming theorem. It states the universal mapping properties of monomorphisms and epimorphisms. Further considerations are made in [2, Appendix 3, Types of Morphisms].

Lemma 1.5 (Universal Mapping Property of Injective and Surjective Maps)**.** *Let $f: A \to B$ be a set map.*

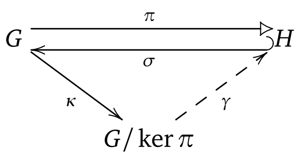
(i) *The map f is injective if and only if for all sets T and maps $g, h\colon T \to A$ with $g \circ f = h \circ f$ we have $g = h$.*



(ii) *The map f is surjective if and only if for all sets T and maps $g, h\colon B \to T$ with $f \circ g = f \circ h$ we have $g = h$.*



Theorem 1.6 (Split Isomorphism Theorem for Topological Groups)**.** *We consider topological groups G and H with a continuous group morphism $\pi\colon G \to H$ and a continuous section $\sigma\colon H \to G$ of π, i.e. $\sigma \circ \pi = \mathrm{id}_H$. Then there is a uniquely determined morphism $\gamma\colon G/\ker\pi \to H$ such that $\pi = \kappa \circ \gamma$. Furthermore, σ is an embedding and π is a quotient map. Letting $\kappa\colon G \to G/\ker\pi$ denote the natural projection then yields the following commutative diagram:*



Proof. By the homomorphism theorem 1.3 for topological groups there exists a unique continuous group isomorphism $\gamma\colon G/\ker\pi \to H$ such that $\pi = \kappa \circ \gamma$.

Next, we show $\gamma = (\sigma \circ \kappa)^{-1}$. By the section property of σ we obtain $(\sigma\kappa)\gamma = \sigma\pi = \mathrm{id}_H$. From this we conclude $\kappa\gamma = \kappa\gamma(\sigma\kappa\gamma)$. Invoking Lemma 1.5, by the injectivity of γ we have $\kappa = \kappa(\gamma\sigma\kappa)$ and by the surjectivity of κ it follows that $\mathrm{id}_{G/\ker\pi} = \gamma(\sigma\kappa)$.

Hence, $\gamma^{-1} = \sigma \circ \kappa$ is also continuous, so γ is a homeomorphism. Since additionally $\pi = \kappa \circ \gamma$ we find that π is a quotient map.

We denote the corestriction of σ to its image by $\tau\colon H \to H^\sigma$. Because σ is a section of π, we have $\tau \circ \pi|_{H^\sigma} = \mathrm{id}_H$ and $\pi|_{H^\sigma} \circ \sigma = \mathrm{id}_{H^\sigma}$, so τ is a bijective map with continuous and open inverse $\tau^{-1} = \pi|_{H^\sigma}\colon H^\sigma \to H$. Thus, τ is continuous and σ is an embedding, i.e. a homeomorphism onto its image. □

Lemma 1.7. *Let $\pi\colon G \to Q$ be a quotient morphism of topological groups with kernel N. Then π is an open map.*

Proof. Let $U \subset G$ be open. Then $UN = \bigcup_{x \in N} Ux$ is also open. By the property of the quotient topology, U^π is open iff $U^{\pi\pi^{-1}}$ is open. As $UN = U^{\pi\pi^{-1}}$, the claim follows. □

Theorem 1.8. *Let $\pi\colon G \to Q$ be a quotient morphism of topological groups with $N := \ker\pi \trianglelefteq G$. If $H \leq G$, then*

$$\alpha\colon NH/N \to H^\pi,\ Nh \mapsto h^\pi$$

is an isomorphism of topological groups and

$$\beta\colon H/(N \cap H) \to NH/N,\ (N \cap H)h \mapsto Nh$$

is a bijective morphism of topological groups.

Proof. We denote the natural projections by $\pi_{N\cap H}\colon H \to H/(N \cap H)$ and by $\pi_N\colon NH \to NH/N$. Then, by the well-known second isomorphism theorem for groups, there exists the desired bijective group homomorphism $\beta\colon H/(N\cap H) \to NH/N$. If $\gamma\colon H \hookrightarrow NH$ is the inclusion, then $\pi_{N\cap H} \circ \beta = \gamma \circ \pi_N$ and the following diagram commutes:

$$
\begin{array}{ccccccc}
H & \overset{\gamma}{\hookrightarrow} & NH & \hookrightarrow & & & G \\
\downarrow{\scriptstyle \pi_{N\cap H}} & & \downarrow{\scriptstyle \pi_N} & & & & \downarrow{\scriptstyle \pi} \\
H/(N\cap H) & \underset{\beta}{\longrightarrow} & NH/N & \underset{\alpha}{\longrightarrow} & H^{\pi} & \hookrightarrow & Q
\end{array}
$$

As the composition $\pi_{N\cap H} \circ \beta = \gamma \circ \pi_N$ is continuous and $\pi_{N\cap H}$ is a quotient map, we can apply Lemma 1.2 which shows the continuity of β.

Indeed, α is well-defined, since $Nh = N\tilde{h} \in NH/N$ implies $h\tilde{h}^{-1} \in N = \ker\pi$, i.e. $h^{\pi} = \tilde{h}^{\pi}$. Straightforward calculation shows that it is a homomorphism, too.

Let $\varphi := \pi|_{NH} \colon NH \to H^{\pi}$. From $\operatorname{im}\varphi = \operatorname{im}(\pi_N \circ \alpha) = H^{\varphi} = H^{\pi}$ we conclude $\operatorname{im}\alpha = H^{\pi}$, so α is surjective. We now assume α were not injective. Then there would be an $Nh \in NH/N$ such that $Nh \neq N$ and $Nh^{\alpha} = 1_Q \in H^{\pi}$. Consequently, we would have $h \in H \cap N^{\complement}$ and $h^{\pi} = 1_Q$ contradicting the fact $\ker\pi = N$.

The restriction $\varphi = \pi|_{NH}$ is continuous. As π_N is a quotient map, Lemma 1.2 implies the continuity of α. To see that α is open, let $U \subset NH/N$ be an open set. Then $U^{\pi^{-1}}$ is open. Lemma 1.7 implies that $\varphi = \pi|_{NH}$ is open, so $U^{\pi^{-1}\varphi} = U^{\alpha}$ is an open set. In sum, α is therefore an isomorphism of topological groups. □

Theorem 1.9 (Open Mapping Theorem). *We consider a locally compact and σ-compact group G. If H is a locally compact Hausdorff group, then every surjective morphism $G \to H$ is open.*

Proof. In the proof, we treat two distinct cases. If φ is injective, we only need to show that its inverse is continuous. Using the σ-compactness of G and arguing that H is not meager gives for every identity neighborhood in G an identity neighborhood in H whose φ^{-1}-image lies in the fixed neighborhood in G. Assuming that φ is not injective, we can do the proof similarly for the canonical mapping induced via the isomorphism theorem which already suffices for the openness of φ.

Let $G = \bigcup_{n\in\mathbb{N}} C_n$ be a locally compact group and $(C_n)_{n\in\mathbb{N}}$ a countable family of compact subsets of G. For a locally compact Hausdorff group H, we consider a surjective morphism $\varphi \colon G \to H$ between topological groups.

We at first assume φ to be injective, hence bijective. Then it suffices to show the continuity of φ^{-1}, i.e. continuity at 1_H. In order to do so, We consider an identity neighborhood $U \subset G$. Using the local compactness of G there is a compact identity neighborhood $V \subset U$. By [3, Lemma 3.21], we can choose V such that $V = V^{-1}$ and $V \cdot V \subset U$. For each $n \in \mathbb{N}$ we have an open cover of C_n by $C_n \subset C_n \cdot V^{\circ} = \bigcup_{c\in C_n} cV^{\circ}$. As the sets C_n are compact, we find for each $n \in \mathbb{N}$ a finite subset $F_n \subset C_n$ such that $C_n \subset F_n V^{\circ}$. Then $G = \bigcup_{n\in\mathbb{N}} F_n V$ and therefore $H = \bigcup_{n\in\mathbb{N}} (F_n V)^{\varphi}$. We define $A := \bigcup_{n\in\mathbb{N}} (F_n)^{\varphi}$. As F_n is finite, the image $(F_n)^{\varphi}$ is also finite. Thus, A is countable. The locally compact Hausdorff space $H = \bigcup_{a\in A} a \cdot (V)^{\varphi}$ is not meager by [3, Lemma 1.29]. We can therefore choose a set $a \cdot (V)^{\varphi}$, $a \in A$, with non-empty interior which also implies $(V^{\varphi})^{\circ} \neq \emptyset$. Hence, for some $v \in V$ the set V^{φ} is a neighborhood of v^{φ} in H. Then $W := V^{\varphi} \cdot V^{\varphi}$ is an identity neighborhood in H with $W^{\varphi^{-1}} = V \cdot V \subset U$.

Finally we consider the case that φ is not injective. Let $\pi \colon G \to G/\ker\varphi$ be the canonical projection. Then by the isomorphism theorem 1.3 there is a bijective continuous group morphism $\beta \colon G/\ker\varphi \to H$ such that $\varphi = \pi \circ \beta$. Now, $G/\ker\varphi$ is also locally compact as shown in [3, Theorem 6.7/(b)]. Furthermore, $G^{\pi} = G/\ker\varphi$ is σ-compact. So we analogously can do the above proof replacing G by the quotient $G/\ker\varphi$ and $\varphi \colon G \to H$ by the induced map $\beta \colon G/\ker\varphi \to H$. Thus β is open and consequently also φ is. □

Example 1.10. The exponential morphism $\mathbb{R} \to \mathbb{T}$, $t \mapsto \exp(2\pi i t)$ is an open map, since $\mathbb{R} = \bigcup_{n\in\mathbb{N}} [-n, n]$ is σ-compact. See also Example 1.4.

Next we consider two cases of bijective topological group homorphisms which are not open, hence no isomorphisms.

Example 1.11. The following examples show that both of the conditions that G is σ-compact or H is Hausdorff are crucial.

We denote the discrete topology by $\mathscr{D}$ and the indiscrete topology by $\mathscr{U}$. Clearly, $(\mathbb{R}, \mathscr{U})$ is locally compact. Also, $(\mathbb{R}, \mathscr{D})$ is locally compact, as for any $x \in (\mathbb{R}, \mathscr{D})$ the singleton $\{x\}$ is an open finite set, thus compact.

But $(\mathbb{R}, \mathscr{D})$ is not σ-compact. Compact spaces with respect to $\mathscr{D}$ are finite and countable unions of finite sets are countable, but $\mathbb{R}$ is uncountable.

The set-theoretical identities $(\mathbb{R}, \mathscr{D}) \to (\mathbb{R}, \mathscr{U})$ and $(\mathbb{R}, \mathscr{U}) \to (\mathbb{R}, \mathscr{D})$ are continuous group isomorphisms, but not open, hence, no isomorphisms of topological groups.

Example 1.12. Let F be a nontrivial finite group. If c is an infinite cardinal number, we consider the product $P := F^c$ which can be identified with $\prod_{i\in I} F$ for an arbitrary index set I of cardinality c. Then

$$P \times P = \prod_{i\in I} F \times \prod_{i\in I} F \cong \prod_{i\in I \coprod I} F = F^{c+c}$$

as groups. From set theory it is known that $c + c = c$ as c is infinite, so overall

$$P \times P = F^c \times F^c \cong F^{c+c} = F^c = P.$$

Moreover, these identifications respect the discrete topology $\mathscr{D}$ as well as the product topology $\mathscr{P}$, so we have isomorphisms $\alpha\colon (P,\mathscr{D})^2 \to (P,\mathscr{D})$ and $(P,\mathscr{P})^2 \to (P,\mathscr{P})$ in the sense of topological groups. Changing the second factor from $(P,\mathscr{P})$ into $(P,\mathscr{D})$ we obtain a new map, which is again continuous. Calling this morphism β, we define

$$\gamma := \alpha^{-1} \times \beta\colon (P,\mathscr{D}) \times (P,\mathscr{D}) \times (P,\mathscr{P}) \to (P,\mathscr{D}) \times (P,\mathscr{D}) \times (P,\mathscr{P})$$

i.e.

$$((u,v)^{\alpha}, y, z)^{\gamma} = (u, v, (y,z)^{\beta})$$

for $u, v, y, z \in \mathscr{P}$. This is easily seen to be a bijective endomorphism of topological groups. But, since in general the discrete topology $\mathscr{D}$ is actually finer than $\mathscr{P}$, the inverse β^{-1} is not continuous. Thus, the inverse group morphism $\gamma^{-1} = \alpha \times \beta^{-1}$ is not continuous either, so γ is not a homeomorphism. In particular, it is a continuous bijective group endomorphism, but not an automorphism of topological groups.

Theorem 1.13. *Let G be a topological groups with subgroups $A, B \leq G$ such that $AB = G$ and $A \cap B = \{1\}$. Then we have the following:*

(i) *The multiplication*

$$\varphi\colon A \times B \to G,\ (a,b) \mapsto a \cdot b$$

is a continuous bijective map.

(ii) *The map φ is a group homomorphism iff both A and B are normal in G.*

(iii) *If B is closed and normal in G, A and G/B are locally compact and A is σ-compact, then φ is a homeomorphism.*

Proof. (i) From $(A \times B)^{\varphi} = AB = G$ we deduce, that φ is surjective. Then, φ is simply the group multiplication map which is continuous by construction of the group topology on G. We only have to show injectivity. Let $(a,b)^{\varphi} = (a',b')^{\varphi}$ for $a, a' \in A$ and $b, b' \in B$, i.e. $ab = a'b'$. This is equivalent to $\underbrace{a'^{-1}a}_{\in A} = \underbrace{b'b^{-1}}_{\in B}$ which implies $a'^{-1}a \in A \cap B = \{1\}$. Thus, $a = a'$ and analogously $b = b'$.

(ii) Let $A, B \trianglelefteq G$ and $x := a^{-1}b^{-1}ab \in G$. From $b^{-1}ab \in A$ it follows that $a^{-1}(b^{-1}ab) = x \in A$, too. Similarly, we have $x \in B$, i.e. $x \in A \cap B = \{1\}$. Therefore, G is Abelian, so for all $(a,b), (a',b') \in A \times B$ it follows

$$\varphi((a,b)\cdot(a',b')) = \varphi(aa', bb') = (aa')(bb') = aba'b' = \varphi(a,b)\varphi(a',b')$$

as desired.
Conversely, if φ is a group homomorphism, then it is a group isomorphism by (i). As $A \times \{1\}$ is the kernel of the second coordinate projection $A \times B \to B$, the subgroup $A \times \{1\}$ is normal in $A \times B$. Thus, $A = (A \times \{1\})^{\varphi}$ is normal in G as the image of a normal subgroup under the surjective morphism φ. Analogously, we obtain $B \trianglelefteq G$.

(iii) If $\pi\colon G \to G/B$ is the canonical projection, its restriction $\psi := \pi|_A\colon A \to G/B = (AB)/B$ is a group morphism. Surjectivity is clear, because for $Bab \in G/B$ we simply calculate $Bab = Ba \cdot Bb = Ba \cdot B = Ba = a^{\psi}$. Thus, to prove injectivity, we consider $Ba = Ba'$ for some $a, a' \in A$, i.e. $aa'^{-1} \in B$. Triviality of $A \cap B$ yields $a = a'$. Because of [3, Proposition 6.6], by the closedness of B the quotient G/B is a Hausdorff space. Therefore, we can apply the open mapping theorem and observe that ψ is also open, hence an isomorphism of topological groups.
Then $\pi \circ \psi^{-1}$ is continuous. Obviously for $g = ab \in G$ where $a \in A$ and $b \in B$, we have $g^{\pi\psi^{-1}} = (a^{\pi})^{\psi^{-1}} \cdot B^{\psi^{-1}} = a$. This leads to

$$g^{\varphi^{-1}} = \left(g^{\pi\psi^{-1}}, \left(g^{\pi\psi^{-1}}\right)^{-1} g\right) \in A \times B$$

which implies that φ^{-1} is continuous as tupling and composing of functions preserves continuity. Hence φ is open. □

Definition 1.14 (Interior Direct Product of Topological Groups)**.** Let G be a topological group with normal subgroups A and B such that $AB = G$ and $A \cap B = \{1\}$. If the multiplication map of the preceding theorem is an isomorphism of topological groups, then $G \cong A \times B$. In this case, G is said to be the *interior direct group of A and B*, written

$$G = A \oplus B.$$

Example 1.15. Investigating the multiplicative group $G := \mathbb{C}^\times$ implies for $A := \mathbb{R}_{>0}$ and $B := \mathbb{T}$ that $G = AB = \mathbb{R}_{>0}\mathbb{T}$ and $A \cap B = \mathbb{T} \cap \mathbb{R}_{>0} = \{1\}$. Then $\mathbb{C}^\times \cong \mathbb{R}_{>0} \oplus \mathbb{T}$ via the multiplication map.

Example 1.16. Let $A := \mathbb{Z}$, $B := r\mathbb{Z}$ for a number $r \in \mathbb{R} \setminus \mathbb{Q}$ and $G := A + B = \mathbb{Z} + r\mathbb{Z} \leq \mathbb{R}$. Then $A = \mathbb{Z}$ and $B = r\mathbb{Z}$ are discrete in $\mathbb{R}$ so $A \times B = \mathbb{Z} \times r\mathbb{Z}$ is discrete as well, but G is not discrete.

2 Cyclic Subgroups

Theorem 2.1 (Weil's Lemma)**.** *Let G be a locally compact Hausdorff group and $H = \mathbb{Z}$ or $H = \mathbb{R}$. If $\varphi\colon H \to G$ is a morphism of topological groups, either φ induces an isomorphism $H \to H^\varphi$ or $\overline{H^\varphi}$ is compact in G.*

Proof. In case $H = \mathbb{Z}$ it is not necessary to require φ to be continuous, since continuity follows from the fact that we consider $\mathbb{Z}$ under the discrete topology. In the upcoming calculations, we only have to focus on the set $G' := \overline{H^\varphi}$, thus we replace G by G' and φ by its corestriction on G', also denoted by φ. Then, G' is a commutative subgroup of G by [3, Lemma 4.4].

In the following, we on one side consider the case that φ is not injective. Then G' is compact by a homomorphism theorem argument. On the other side, if φ is not injective, we distinguish between the case when there is a bounded neighborhood in G' such that its preimage under φ is bounded in H or, contrarily, all the neighborhoods have unbounded preimages. In the first case we apply the open mapping theorem. Then, in fact $H \cong H^\varphi$ as topological groups. To prove the unbounded case, we derive the compactness of G' by a computational argument for a suitbale compact identity neighborhood.

For the sake of simplification, in the following, for a real interval I we will denote the set $I \cap H$ by I, also if $H = \mathbb{Z}$ and hence, $I \cap H$ is not an interval.

Now, assuming φ is not injective, we pick a $k \in \ker\varphi \setminus \{0\}$. If $\pi\colon H \to H/\langle k\rangle$ is the canonical projection, by the homomorphism theorem φ induces the bijective continuous group morphism $\beta\colon H/\langle k\rangle \to G'$ such that $\varphi = \pi \circ \beta$. As $H/\langle k\rangle = H^\pi$ is compact, also $H^{\pi\circ\beta} = H^\varphi = G'$ is.

We henceforth take φ to be injective. By definition, H^φ lies densely in G', therefore every nonempty neighborhood in G' has a nonempty preimage in H under φ.

If there is a neighborhood U in G' so $U^{\varphi^{-1}} \subset H$ is bounded, then $\overline{U^{\varphi^{-1}}} =: C$ is compact by the Heine-Borel theorem. Obviously, $x \in U^{\varphi^{-1}\varphi}$ iff there exists an $h \in H$ such that $x = h^\varphi \in U$ which finally is equivalent to $x \in U \cap H^\varphi$. Thus, $U \cap H^\varphi = U^{\varphi^{-1}\varphi}$. As the compact image C^φ is closed in the hausdorff space H^φ, the set $U^{\varphi^{-1}\varphi}$ has compact closure $\overline{U^{\varphi^{-1}\varphi}} = \overline{C^\varphi} = C^\varphi$. By the homogenity of the topological group H^φ, we find for every point in H^φ a relatively compact neighborhood, thus H^φ is locally compact. This allows us to apply the open mapping theorem 1.9, so $\varphi\colon H \to H^\varphi$ is an open map. Also, φ is assumed to be an injective topological group morphism, thus altogether it is an isomorphism $H \to H^\varphi$.

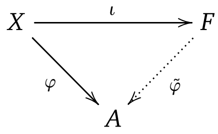
Now let each neighborhood in G' have an unbounded φ-preimage. We absurdly assume there were a neighborhood $V \subset G'$ whose preimage were bounded above. Then we could pick a $t \in H$ with $t^\varphi \in V^\circ$ and $2t^\varphi - V \in \mathscr{U}(t^\varphi)$ would have a φ-preimage which would be bounded below. Altogether, $(V \cap (2t^\varphi - V))^{\varphi^{-1}}$ would be bounded in contradiction to the premise that V^φ is unbounded. Hence, for all $h \in H$ and neighborhoods $V \subset G'$ we find $[h,\infty)^\varphi \cap V \neq \emptyset$ which means the image $[h,\infty)^\varphi$ is dense in G'. Let $\mathscr{B}(0)$ be an identity neighborhood basis in G' and $U \subset G'$ be a compact identity neighborhood. By [3, Lemma 3.17], we have the representation

$$G' = \overline{[h,\infty)^\varphi} = \bigcap_{V\in\mathscr{B}(0)} V + [h,\infty)^\varphi \subset U^\circ + [h,\infty)^\varphi.$$

Of course, also the inclusion $U^\circ + [h,\infty)^\varphi \subset G'$ holds, thus these two sets are equal.

As U is compact and the family $(U + x)_{x\in(0,\infty)}$ admits an open covering, there exists a finite subset $F \subset (0,\infty)$ such that $U \subset U + F^\varphi$. Consequently, F has a maximal element, say $m := \max F$. Again, from the compactness of U we derive that for an arbitrary $g \in G'$, also $U + g$ is compact. In particular, the set $[0,\infty) \cap (U + g)^{\varphi^{-1}}$ has a smallest element s. Now, $s^\varphi - g \in U$ implies that there exists $f \in F$ such that $s^\varphi - g \in U + f^\varphi$, i.e. $(s - f)^\varphi \in U + g$. The minimality of s leads to $s - f < 0$. Then $s < f \leq m$ and thus $g \in s^\varphi - U \subset [0,m]^\varphi - U$. The value m is independent from g, so $G' \subset [0,m]^\varphi - U$ is compact. ☐

Definition 2.2 (Free Abelian Group)**.** Let F be an Abelian group. Then F is said to be *free Abelian (of rank c)* if there exist a set X of cardinality c and a map $\iota : X \to F$ so the following universal property holds: For every Abelian group A and set map $\varphi : X \to A$ there is a unique group morphism $\tilde{\varphi} : F \to A$ such that $\varphi = \iota \circ \tilde{\varphi}$, i.e. the following diagram commutes:



Lemma 2.3. *For a free Abelian group F, the map $\iota : X \to F$ is injective.*

Proof. Given an injective map $\varphi : X \to A$ where A is an Abelian group, we obtain a unique morphism $\tilde{\varphi} : F \to A$ such that $\varphi = \iota \circ \tilde{\varphi}$. Assuming ι were not injective, we would find $x, y \in X$ such that $x \neq y$ and $x^\iota = y^\iota$. Then $x^{\iota\circ\tilde{\varphi}} = y^{\iota\circ\tilde{\varphi}}$, i.e. $x^\varphi = y^\varphi$ contradicting the injectivity of φ. □

The preceding lemma justifies the identification of X with its image under the inclusion $X^\iota \subset F$. The following instructive example in this manner shows X as a "basis" of F similiarly to the concept in Linear Algebra.

Example 2.4. For every natural number $n \in \mathbb{N}$ the group $\mathbb{Z}^n$ is free Abelian of rank n. To see this, we define $X := \{1, \ldots, n\}$ and the injection $\iota : X \hookrightarrow \mathbb{Z}^n$ by $\iota(i) := (0, \ldots, 1, \ldots, 0)$ with 1 at the i-th position and 0 elsewhere. Given a map $\varphi : X \to A$ where A is an Abelian group, there is obviously a unique one group morphism $\tilde{\varphi} : \mathbb{Z}^n \to A$ satisfying $\varphi = \iota \circ \tilde{\varphi}$. It is defined by

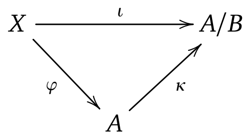
$$(z_1, \ldots, z^n)^{\tilde{\varphi}} := \sum_{i=1}^{n} z_i i^\varphi$$

where multiplication by z_i means the z_i-fold sum.

Lemma 2.5. *Let A be an Abelian topological group and $B \leq A$ an open subgroup such that A/B is a free Abelian topological group. Then there exists a is a discrete subgroup $C \leq A$ such that $A \cong B \oplus C$ and $C \cong A/B$ as topological groups.*

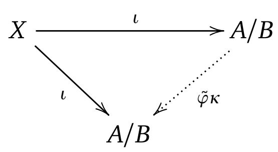
Proof. The idea is to obtain a morphism by the universal mapping property of the free Abelian group who embeds A/B isomorphically into A.

Consequenty, as A/B is free Abelian, there exist a set X and an injection $\iota : X \to A/B$. We now define a map $\varphi : X \to A$ making



commute by choosing for every $x \in X$ an element x^φ such that $x^\iota = B + x^\varphi$. Well-definedness of φ follows from the injectivity of ι. To see this, we assume $x \neq \tilde{x}$ for $x, \tilde{x}$ in X. Then $x^\iota \neq \tilde{x}^\iota$, i.e. we can choose elements $a, \tilde{a} \in A$ such that $a^\kappa = B + a = x^\iota \neq \tilde{x}^\iota = B + \tilde{a} = \tilde{a}^\kappa$. Setting $a := x^\varphi$ and $\tilde{a} := \tilde{x}^\varphi$ yields the desired mapping.

Considering the diagram above, we have $\iota(\tilde{\varphi}\kappa) = \varphi\kappa = \iota$. Hence, $\tilde{\varphi}\kappa$ makes the diagram



commute, so it must equal the morphism from the universal property, i.e. $\tilde{\varphi}\kappa = \tilde{\iota} = \mathrm{id}_{A/B}$.

We set $C := F^{\tilde{\varphi}} = (A/B)^{\tilde{\varphi}}$. Then $\ker\kappa = B$ and $\operatorname{im}\kappa = A/B$ imply $A^\kappa = B^\kappa + F^\kappa = (B + (A/B)^{\tilde{\varphi}})^\kappa$. Considering the preimages we find $A = B + C$. Now, we establish the injectivity of the restricted projection $\kappa|_C : C \to A/B$. Taking $c, c' \in C = F^{\tilde{\varphi}}$ where $c \neq c'$ means, there exist $a, a' \in A/B$ where $c = (B + a)^{\tilde{\varphi}}$ and $c' = (B + a')^{\tilde{\varphi}}$ while $a^\kappa \neq (a')^\kappa$. The above observation $\tilde{\varphi}\kappa = \mathrm{id}_{A/B}$ leads to $c^\kappa = (B + a)^{\tilde{\varphi}\kappa} = B + a \neq B + a' = (B + a')^{\tilde{\varphi}\kappa} = (c')^\kappa$, which shows the injectivity of $\kappa|_C$ and $B \cap C = \ker(\kappa|_C) = \{0\}$.

Finally, combining the results $A = B + C$ and $B \cap C = \{0\}$, we apply theorem 1.13 and establish that

$$\psi : B \times C \to B + C,\ (b, c) \mapsto b + c$$

is a bijective morphism of topological groups.

□

Corollary 2.6. *Let A be an Abelian topological group. If $B \leq A$ is subgroup such that A/B is discrete and isomorphic to $\mathbb{Z}^n$ for some $n \in \mathbb{Z}$, then $A \cong B \times \mathbb{Z}^n$ as topological groups.*

Proof. Obviously, the preceding theorem can be applied for $C = \mathbb{Z}^n$. □

Theorem 2.7. *Let A be an Abelian locally compact Hausdorff group. If $V \in \mathcal{U}(0_A)$ is compact in A and $B := \langle V \rangle \leq A$, then there exists a discrete subgroup $D \leq B$ such that B/D is compact and $D \cap V = \{0\}$. In particular, we can choose $D \cong \mathbb{Z}^d$ for some $d \in \mathbb{N}$.*

Proof. To prove the claim, we decompose B into the sum of a compact set and a finitely generated group C. The desired group D turns out to be a particular subgroup of C. Arguing that the image of C under the canonical projection $B \to B/D$ has no infinite cyclic subgroup, we show that its image is dense which leads to the compactness of B/D.

We choose a compact identity neighborhood V and consider the subgroup $B := \langle V \rangle$ generated by it. Then there exists an open set $O \subset V \subset B$ and $O + B$ is open as the translate of B by an open set. By $B \subset O + B \subset B + B = B$ we tell that $B = O + B$ again is open. Setting $W := V \cup (-V)$ we gain a compact identity neighborhood, hence $B = \langle W \rangle$ is locally compact and $W = -W$. We recursively define

$$W_0 := \{0\}$$
$$W_{n+1} := W_n + W$$

for $n \in \mathbb{N}$. Then B is countably covered by $B = \bigcup_{n \in \mathbb{N}} W_n$. As $W_2 = W + W$ is compact, we find a finite subset $F \subset B$ such that $W_2 \subset F + W^\circ \subset F + W$. Putting $C := \langle F \rangle$ we derive

$$W_1 \subset W_2 \subset \langle F \rangle + W = C + W.$$

As $W_n \subset C + W$ implies

$$W_{n+1} = W_n + W \subset C + (W + W) = C + W_2 = C + W,$$

the above yields $B = \bigcup_{n \in \mathbb{N}} W_n \subset C + W$, i.e. $B = C + W$.

Now, C is finitely generated. This means, every element in C can be written as a finite linear combination of generating elements with integer coefficients. Put another way, the set

$$\{n \in \mathbb{N} \colon \text{There exists an injective morphism } \iota \colon \mathbb{Z}^n \to C.\}$$

is bounded. Thus, there is a maximal natural number $d \in \mathbb{N}$ such that there exists a discrete subgroup $D \leq C$ with $D \cong \mathbb{Z}^d$. Then $D \cap W$ is finite as a discrete subset of a compact set. Considering $D' := mD$ for a sufficiently large $m \in \mathbb{N}$, we obtain $\{0\} = D' \cap W \supset D' \cap V$.

Considering the canonical projection $\pi \colon B \to B/D'$, we claim C^π does not contain any discrete infinite cyclic group. If this were wrong, we could find a discrete subgroup of C isomorphic to $\mathbb{Z}^{d+1}$ in contradiction to the minimality of d. Too see the compactness of $\overline{C^\pi}$, we initially remark that for any $c \in C$ the cyclic group is either finite or non-discrete because of [3, Corollary 3.13]. From Weil's Lemma 2.1 we derive that $\langle c^\varphi \rangle$ is relatively compact in B/D, therefore $\overline{C^\pi} = \sum_{f \in F} \overline{\langle f^\pi \rangle}$ is compact.

Eventually, $B/D = C^\pi + W^\pi = \overline{C^\pi} + W^\pi$ is compact. □

Finally, we prove a useful characterization of finite cyclic groups. As a corollary we show that for a commutative field F, every finite subgroup of the group of units $F^\times$ is cyclic.

Definition 2.8. The map

$$\varphi \colon \mathbb{N} \to \mathbb{N},\ n \mapsto |\{a \in \mathbb{Z}/n\mathbb{Z} \colon \operatorname{ord}(a) = d\}|$$

is called *Euler's totient function* or *φ-Function*.

The proof of the following theorem can be found in many texbooks, i.e. [1, 4.5, Remark 4].

Theorem 2.9 (Totient Function and Generators of a Cyclic Group)**.** *The number of the generators of cyclic group of order d is given by*

$$\varphi(d) = |(\mathbb{Z}/d\mathbb{Z})^\times|.$$

Theorem 2.10. *A finite group G with $\operatorname{ord} G = n$ is cyclic iff for every divisor $d|n$ there is at most one subgroup of order d.*

Proof. Let G be cyclic. If there exist $H, H' \leq G$ of order d for $d|n$, then $H \cong \mathbb{Z}/d\mathbb{Z} \cong H'$ by the classification theorem for cyclic groups, cf. [1, 1.3, Theorem 3].

To prove the converse statement, we consider an element $a \in G$ of order $d|n$. Then, every other element $b \in G$ of order d is contained in $\langle a \rangle$. We define a function $f : \mathbb{N} \to \mathbb{N}_0$ where $f(k)$ is the number of all elements of order k in G. Then, $\{b \in G\colon \operatorname{ord}(b) = d\} \subset \langle a \rangle$ implies $f(d) \leq \varphi(d)$. On the other hand, by the preceding theorem, we have

$$\begin{aligned} n = |G| = \left| \coprod_{d|n} \{a \in G\colon \operatorname{ord} a = d\} \right| = \sum_{d|n} f(d) \\ \leq \sum_{d|n} \varphi(d) = \left| \coprod_{d|n} \{a \in \mathbb{Z}/d\mathbb{Z}\colon \operatorname{ord} a = d\} \right| = |\mathbb{Z}/n\mathbb{Z}| = n. \end{aligned}$$

Then

$$\sum_{d|n} \underbrace{\varphi(d) - f(d)}_{\geq 0} = 0,$$

so $\varphi(d) = f(d)$ for all $d|n$. In particular, $f(n) = \varphi(n) > 0$, thus there exists an element $a \in G$ of order n, so necessarily $G = \langle a \rangle$ is cyclic. □

Corollary 2.11. *Let F be a commutative field. Then every finite subgroup of the group of units $F^\times$ is cyclic.*

Proof. Let $n := \operatorname{ord} F^\times$ and $G \leq F^\times$ be a subgroup of order $d|n$. Then, for all elements $g \in G$ we have $\operatorname{ord} g | d$, so g is a zero of the polynomial $f := X^d - 1 \in F[X]$. As f has at most d distinct zeros in F, we conclude, there is no other subgroup of order d in $F^\times$. Thus, invoking Theorem 2.10, $F^\times$ is cyclic. □

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